

# On Non-Gaussian Limiting Laws for the Certain Statistics of the Wigner Matrices

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## Abstract

We continue investigations of our papers [24, 25, 26], in which there were proved CLTs for linear eigenvalue statistics  $\text{Tr } \varphi(M^{(n)})$  and there were found the limiting probability laws for the normalised matrix elements  $\sqrt{n}\varphi_{jj}(M^{(n)})$  of differential functions of real symmetric Wigner matrices  $M^{(n)}$ . Here we consider another spectral characteristic of Wigner matrices,  $\xi_n^A[\varphi] = \text{Tr } \varphi(M^{(n)})A^{(n)}$ , where  $\{A^{(n)}\}_{n=1}^\infty$  is a certain sequence of non-random matrices. We show first that if  $M^{(n)}$  belongs to the Gaussian Orthogonal Ensemble (GOE), then  $\xi_n^{A^\circ}[\varphi]$  satisfies CLT. Then we consider Wigner matrices with i.i.d. entries possessing entire characteristic function and find the limiting probability law for  $\xi_n^{A^\circ}[\varphi]$ , which in general is not Gaussian.

## 1 Introduction

The asymptotic behavior of spectral characteristics of large random matrices  $M^{(n)}$ , when the size  $n$  of matrix tends to infinity, is of the great interest in the random matrix theory. One of the main questions under the study is the validity of CLT. In the last two decades there was obtained a number of results on the CLT for linear eigenvalue statistics  $\text{Tr } \varphi(M^{(n)})$  and other spectral characteristics (see [2, 6, 8, 9, 11, 12, 15, 17, 18, 20, 21, 38, 39, 40, 41, 42] and references therein). It was found that in many cases fluctuations of various spectral characteristics of eigenvalues of random matrix ensembles are asymptotically Gaussian (see [2, 6, 12, 14, 17, 21, 33, 38, 39, 41, 42]). But the CLT is not always the case. Thus it was shown in [33] that the CLT for linear eigenvalue statistics is not necessarily valid for so called hermitian matrix models, for which in certain cases appear non-Gaussian limiting laws.

Another example of non-Gaussian limiting behavior is presented in works [26, 30, 38] dealing with the normalized individual matrix elements  $\sqrt{n}\varphi_{jj}(M^{(n)})$  of functions of real symmetric Wigner random matrix. The particular case of matrix elements  $\sqrt{n}\varphi_{jj}(\widehat{M}^{(n)})$  with  $\widehat{M}^{(n)}$  belonging to the GOE was considered earlier in [26], where it was proved that  $\sqrt{n}(\varphi_{jj}(\widehat{M}^{(n)}))^\circ$  satisfies the CLT. But in [26, 30, 38] it was shown that in general case of Wigner matrices the limiting probability law for  $\sqrt{n}(\varphi_{jj}(M^{(n)}))^\circ$  is not Gaussian but the sum of the Gaussian law and probability law of entries of  $\sqrt{n}M^{(n)}$  modulo a certain rescaling, and to obtain the CLT, one has to impose an integral condition on the test function.

In particular, the fact that in contrast to the linear statistics of eigenvalues, individual matrix elements in general do not satisfy CLT reflects influence of eigenvectors and gives

some information about asymptotic properties of eigenvectors. Indeed, in the case of the Gaussian random matrices (GOE, null Wishart) the eigenvectors are rotationally invariant and according to recent works [5, 13, 22] the eigenvectors of the non-Gaussian random matrices (Wigner, sample covariance) are similar in several aspects to the eigenvectors of the Gaussian random matrices. On the other hand, the results of [25] and [26, 30, 38] imply that there are asymptotic properties of eigenvectors of the non-Gaussian random matrices which are different from those for the Gaussian random matrices.

This paper continues the investigations of [24, 25, 26]. Here we consider random variable

$$\xi_n^A[\varphi] = \text{Tr } \varphi(M^{(n)})A^{(n)}, \quad (1.1)$$

where  $\varphi$  is a smooth enough test-function and  $\{A^{(n)}\}_{n=1}^\infty$  is a sequence of  $n \times n$  non-random matrix satisfying

$$(i) \quad \lim_{n \rightarrow \infty} n^{-1} \text{Tr } A^{(n)T} A^{(n)} = 1, \quad (1.2)$$

$$(ii) \quad \exists \lim_{n \rightarrow \infty} n^{-1} \text{Tr } A^{(n)} = T_A. \quad (1.3)$$

Let us make some examples:

1. *Linear eigenvalue statistics.* If  $A^{(n)} = I^{(n)}$ , then  $T_A = 1$  and

$$\xi_n^A[\varphi] = \text{Tr } \varphi(M^{(n)}). \quad (1.4)$$

2. *Matrix elements.* If  $A_{lm}^{(n)} = \sqrt{n} \delta_{jl} \delta_{jm}$ , then  $T_A = 0$  and

$$\xi_n^A[\varphi] = \sqrt{n} \varphi_{jj}(M^{(n)}). \quad (1.5)$$

3. *Bilinear forms.* If  $A_{lm}^{(n)} = \sqrt{n} \eta_l \eta_m$ , where

$$\eta^{(n)} = (\eta_1^{(n)}, \dots, \eta_n^{(n)})^T, \quad \lim_{n \rightarrow \infty} \sum_{l=1}^n (\eta_l^{(n)})^2 = 1, \quad (1.6)$$

then  $T_A = 0$  and

$$\xi_n^A[\varphi] = \sqrt{n} (\varphi(M^{(n)}) \eta^{(n)}, \eta^{(n)}). \quad (1.7)$$

Here we find the limiting probability law for  $\xi_n^A[\varphi]$  as  $n \rightarrow \infty$ . Our main result is Theorem 5.1 below, where the limiting expression for characteristic function of  $\xi_n^{A_0}[\varphi]$  is given and written through the cumulants of matrix entries and quantities depending on a sequence  $\{A^{(n)}\}_{n=1}^\infty$ . Let us note that the corresponding theorems for linear eigenvalue statistics (1.4) and matrix elements (1.5) of [24, 25, 26, 36] can be obtained from Theorem 5.1 as particular cases (however, under much stronger conditions).

The paper is organized as follows. Section 2 contains definitions, some known facts and technical means used throughout the paper. In Section 3 we consider the case of the Gaussian Orthogonal Ensemble (GOE) and prove CLT for  $\xi_n^A[\varphi]$  (see [25] for the analogous statements for matrix elements). Then we find the limiting variance, Sections 4, and the limiting probability law, Sections 5, for  $\xi_n^A[\varphi]$  for the Wigner matrices. Section 6 contains auxiliary results. We confine ourselves to real symmetric matrices, although our results as well as the main ingredients of proofs remain valid in the hermitian case with natural modifications.

*Convention:* We will use letter  $c$  for an absolute constant that does not depend on  $j$ ,  $k$ , and  $n$ , and may be distinct on different occasions.

## 2 Definitions and Technical Means

To make the paper self-consistent, we present here several definitions and technical facts that will be often used below. We start with the definition of the Wigner real symmetric matrix  $M^{(n)}$ , and put

$$M^{(n)} = n^{-1/2}W^{(n)}, \quad W^{(n)} = \{W_{jk}^{(n)} \in \mathbb{R}, W_{jk}^{(n)} = W_{kj}^{(n)}\}_{j,k=1}^n, \quad (2.1)$$

where  $\{W_{jk}^{(n)}\}_{1 \leq j \leq k \leq n}$  are independent random variables satisfying

$$\mathbf{E}\{W_{jk}^{(n)}\} = 0, \quad \mathbf{E}\{(W_{jk}^{(n)})^2\} = w^2(1 + \delta_{jk}). \quad (2.2)$$

The case of the Gaussian random variables obeying (2.2) corresponds to the GOE (see e.g. [27]):

$$\widehat{M}^{(n)} = n^{-1/2}\widehat{W}^{(n)}, \quad \widehat{W}^{(n)} = \{\widehat{W}_{jk} = \widehat{W}_{kj} \in \mathbb{R}, \widehat{W}_{jk} \in \mathcal{N}(0, w^2(1 + \delta_{jk}))\}_{j,k=1}^n. \quad (2.3)$$

Here for simplicity sake we define Wigner matrix so that first two moments of its entries match those of GOE. It can be shown that if  $\mathbf{E}\{(W_{jj}^{(n)})^2\} = w^2w_2$ , then corresponding expressions for the limiting variance and characteristic function have additional terms proportional to  $(w_2 - 2)$  (see Remarks 4.5 and 5.2).

We will assume in what follows additional conditions on distributions of  $W_{jk}^{(n)}$ , mostly in the form of existence of certain moments of  $W_{jk}^{(n)}$ , whose order will depend on the problem under study.

The next proposition presents certain facts on Gaussian random variables.

**Proposition 2.1** *Let  $\zeta = \{\zeta_l\}_{l=1}^p$  be independent Gaussian random variables of zero mean, and  $\Phi : \mathbb{R}^p \rightarrow \mathbb{C}$  be a differentiable function with polynomially bounded partial derivatives  $\Phi'_l$ ,  $l = 1, \dots, p$ . Then we have*

$$\mathbf{E}\{\zeta_l \Phi(\zeta)\} = \mathbf{E}\{\zeta_l^2\} \mathbf{E}\{\Phi'_l(\zeta)\}, \quad l = 1, \dots, p, \quad (2.4)$$

and

$$\mathbf{Var}\{\Phi(\zeta)\} \leq \sum_{l=1}^p \mathbf{E}\{\zeta_l^2\} \mathbf{E}\{|\Phi'_l(\zeta)|^2\}. \quad (2.5)$$

The first formula is a version of the integration by parts. The second is a version of the Poincaré inequality (see e.g. [7]). Formula (2.4) is a particular case of more general formula. To write it we recall some definitions. If a random variable  $\zeta$  has a finite  $p$ th absolute moment,  $p \geq 1$ , then we have the expansions

$$\mathbf{E}\{e^{it\zeta}\} = \sum_{j=0}^p \frac{\mu_j}{j!} (it)^j + o(t^p), \quad t \rightarrow 0,$$

and

$$\log \mathbf{E}\{e^{it\zeta}\} = \sum_{j=0}^p \frac{\kappa_j}{j!} (it)^j + o(t^p), \quad t \rightarrow 0, \quad (2.6)$$

where "log" denotes the principal branch of logarithm. The coefficients in the expansion of  $\mathbf{E}\{e^{it\zeta}\}$  are the moments  $\{\mu_j\}$  of  $\zeta$ , and the coefficients in the expansion of  $\log \mathbf{E}\{e^{it\zeta}\}$  are the cumulants  $\{\kappa_j\}$  of  $\zeta$ . For small  $j$  one easily expresses  $\kappa_j$  via  $\mu_1, \mu_2, \dots, \mu_j$ . In particular, if  $\mu_1 = 0$ , then

$$\kappa_1 = 0, \quad \kappa_2 = \mu_2 = \mathbf{Var}\{\zeta\}, \quad \kappa_3 = \mu_3, \quad \kappa_4 = \mu_4 - 3\mu_2^2, \dots \quad (2.7)$$

We have [21, 24]:

**Proposition 2.2** (i) *Let  $\zeta$  be a random variable such that  $\mathbf{E}\{|\zeta|^{p+2}\} < \infty$  for a certain non-negative integer  $p$ . Then for any function  $\Phi : \mathbb{R} \rightarrow \mathbb{C}$  of the class  $C^{p+1}$  with bounded partial derivatives  $\Phi^{(l)}$ ,  $l = 1, \dots, p+1$ , we have*

$$\mathbf{E}\{\zeta \Phi(\zeta)\} = \sum_{l=0}^p \frac{\kappa_{l+1}}{l!} \mathbf{E}\{\Phi^{(l)}(\zeta)\} + \varepsilon_p, \quad (2.8)$$

where

$$|\varepsilon_p| \leq C_p \mathbf{E}\{|\zeta|^{p+2}\} \sup_{t \in \mathbb{R}} |\Phi^{(p+1)}(t)|, \quad C_p \leq \frac{1 + (3 + 2p)^{p+2}}{(p+1)!}. \quad (2.9)$$

(ii) *If the characteristic function  $\mathbf{E}\{e^{it|\zeta|}\}$  is entire, and  $\Phi \in C^\infty$ , then*

$$\mathbf{E}\{\zeta \Phi(\zeta)\} = \sum_{l=0}^{\infty} \frac{\kappa_{l+1}}{l!} \mathbf{E}\{\Phi^{(l)}(\zeta)\} \quad (2.10)$$

provided that for some  $a > 0$

$$|\mathbf{E}\{\Phi^{(l)}(\zeta)\}| \leq a^l, \quad (2.11)$$

and for some  $R = ca$ ,  $c > 1$ ,

$$\sum_{l=0}^{\infty} \frac{|\kappa_{l+1}| R^l}{l!} < \infty. \quad (2.12)$$

Here is a simple "interpolation" corollary showing the mechanism of proximity of expectations with respect to the probability law of an arbitrary random variable and the Gaussian random variable with the same first and second moments. Its multivariate version will be often used below.

**Corollary 2.3** *Let  $\widehat{\zeta}$  be the Gaussian random variable, whose first and second moments coincide with those of given random variable  $\zeta$ . Then:*

(i) *We have under conditions of Proposition 2.2 (i):*

$$\mathbf{E}_\zeta\{\Phi(\zeta)\} - \mathbf{E}_{\widehat{\zeta}}\{\Phi(\widehat{\zeta})\} = \sum_{l=2}^p \frac{\kappa_{l+1}}{2l!} \int_0^1 \mathbf{E}\{\Phi^{(l+1)}(\zeta(s))\} s^{(l-1)/2} ds + \varepsilon'_p, \quad (2.13)$$

where the symbols  $\mathbf{E}_\zeta\{\dots\}$  and  $\mathbf{E}_{\widehat{\zeta}}\{\dots\}$  denote the expectation with respect to the probability law of  $\zeta$  and  $\widehat{\zeta}$ ,  $\{\kappa_j\}$  are the cumulants of  $\zeta$ ,  $\mathbf{E}\{\dots\}$  denotes the expectation with respect to the product of probability laws of  $\zeta$  and  $\widehat{\zeta}$ ,

$$\zeta(s) = s^{1/2}\zeta + (1-s)^{1/2}\widehat{\zeta}, \quad 0 \leq s \leq 1, \quad (2.14)$$

$$|\varepsilon'_p| \leq C_p \mathbf{E}\{|\zeta|^{p+2}\} \sup_{t \in \mathbb{R}} |\Phi^{(p+2)}(t)|, \quad (2.15)$$

and  $C_p$  satisfies (2.9).

(ii) We have under conditions of Proposition 2.2 (ii):

$$\mathbf{E}_\zeta\{\Phi(\zeta)\} - \mathbf{E}_{\widehat{\zeta}}\{\Phi(\widehat{\zeta})\} = \sum_{l=2}^{\infty} \frac{\kappa_{l+1}}{2l!} \int_0^1 \mathbf{E}\{\Phi^{(l+1)}(\zeta(s))\} s^{(l-1)/2} ds \quad (2.16)$$

with  $\zeta(s)$  given above.

The next proposition presents simple facts of linear algebra

**Proposition 2.4** *Let  $M$  and  $M'$  be  $n \times n$  matrices, and  $t \in \mathbb{R}$ . Then we have the following:*

(i) the Duhamel formula

$$e^{(M+M')t} = e^{Mt} + \int_0^t e^{M(t-s)} M' e^{(M+M')s} ds, \quad (2.17)$$

(ii) if for a real symmetric  $n \times n$  matrix  $M^{(n)}$  we put

$$U(t) = U^{(n)}(t) := e^{itM^{(n)}}, \quad t \in \mathbb{R}, \quad (2.18)$$

then  $U(t)$  is a symmetric unitary matrix satisfying

$$U(t_1)U(t_2) = U(t_1 + t_2), \quad \|U(t)\| = 1, \quad \sum_{j=1}^n |U_{jk}(t)|^2 = 1, \quad (2.19)$$

(iii) if  $D_{lm} = \partial/\partial M_{lm}$ , then

$$D_{lm}U_{ab}(t) = i\beta_{lm} (U_{al} * U_{bm} + U_{bl} * U_{am})(t), \quad (2.20)$$

where

$$\beta_{lm} = (1 + \delta_{lm})^{-1} = 1 - \delta_{lm}/2, \quad (2.21)$$

the symbol " $*$ " is defined in Proposition 2.5 (ii), and

$$|D_{lm}^p U_{ab}(t)| \leq c'_p |t|^p, \quad c'_p = 2^p/p!, \quad (2.22)$$

(iv) if  $A^{(n)}$  is an  $n \times n$  matrix and  $\xi_n^A(t) = \text{Tr} A^{(n)} U(t)$ , then

$$D_{lm}(A^{(n)}U)_{ab}(t) = i\beta_{lm} ((A^{(n)}U)_{al} * U_{bm} + U_{bl} * (A^{(n)}U)_{am})(t), \quad (2.23)$$

$$D_{lm}\xi_n^A(t) = i\beta_{lm}(U * C^{(n)}U)_{lm}(t), \quad C^{(n)} = A^{(n)} + A^{(n)T}, \quad (2.24)$$

$$D_{lm}^2 \xi_n^A(t) = -\beta_{lm}^2 (U_{ll} * (U * C^{(n)}U)_{mm} + U_{mm} * (U * C^{(n)}U)_{ll} + 2U_{lm} * (U * C^{(n)}U)_{lm})(t), \quad (2.25)$$

$$D_{lm}(U * A^{(n)}U)_{jk}(t) = i\beta_{lm} (U_{jl} * (U * A^{(n)}U)_{mk} + U_{jm} * (U * A^{(n)}U)_{lk} + U_{lk} * (U * A^{(n)}U)_{jk} + U_{mk} * (U * A^{(n)}U)_{jl})(t), \quad (2.26)$$

$$D_{lm}(U * A^{(n)}U)_{lm}(t) = i\beta_{lm} (U_{ll} * (U * A^{(n)}U)_{mm} + U_{mm} * (U * A^{(n)}U)_{ll} + 2U_{lm} * (U * A^{(n)}U)_{lm})(t). \quad (2.27)$$

It follows from the above that if  $A^{(n)}$  satisfy (1.2) – (1.3), and

$$C_A : \text{Tr } A^{(n)T} A^{(n)} \leq C_A n, \quad \forall n \in \mathbb{N}, \quad (2.28)$$

then

$$|(A^{(n)} U^{(n)})_{lm}| \leq (A^{(n)T} A^{(n)})_{ll}^{1/2} \leq O(n^{1/2}), \quad (2.29)$$

$$|(U^{(n)} A^{(n)} U^{(n)})_{lm}| \leq (\text{Tr } A^{(n)T} A^{(n)})^{1/2} \leq C_A n^{1/2}, \quad (2.30)$$

$$\sum_{l,m=1}^n |(U^{(n)} A^{(n)} U^{(n)})_{lm}|^2 = \text{Tr } A^{(n)T} A^{(n)} = O(n), \quad (2.31)$$

$$\sum_{m=1}^n |(U^{(n)} A^{(n)} U^{(n)})_{mm}|^2 \leq O(n), \quad (2.32)$$

and

$$|\xi_n^A(t)| \leq (n \text{Tr } A^{(n)T} A^{(n)})^{1/2} = O(n), \quad (2.33)$$

$$|D_{lm}^p \xi_n^A(t)| \leq n^{1/2} c_p |t|^p, \quad c_p = C_A 2^{p+1}/p!, \quad (2.34)$$

as  $n \rightarrow \infty$ .

At last we need the generalized Fourier transform, in fact the  $\pi/2$  rotated Laplace transform (see e.g. [44], Sections 1.8-9 for its definition).

**Proposition 2.5** *Let  $f : \mathbb{R}_+ \rightarrow \mathbb{C}$  be a locally Lipschitzian and such that for some  $\delta > 0$*

$$\sup_{t \geq 0} e^{-\delta t} |f(t)| < \infty, \quad (2.35)$$

*and let  $\tilde{f} : \{z \in \mathbb{C} : \Im z < -\delta\} \rightarrow \mathbb{C}$  be its generalized Fourier transform*

$$\tilde{f}(z) = i^{-1} \int_0^\infty e^{-izt} f(t) dt. \quad (2.36)$$

*The inversion formula is given by*

$$f(t) = \frac{i}{2\pi} \int_L e^{izt} \tilde{f}(z) dz, \quad t \geq 0, \quad (2.37)$$

*where  $L = (-\infty - i\varepsilon, \infty - i\varepsilon)$ ,  $\varepsilon > \delta$ , and the principal value of the integral at infinity is used.*

*Denote for the moment the correspondence between functions and their generalized Fourier transforms as  $f \leftrightarrow \tilde{f}$ . Then we have:*

$$(i) \quad \int_0^t f(\tau) d\tau \leftrightarrow (iz)^{-1} \tilde{f}(z);$$

$$(ii) \quad \int_0^t f_1(t - \tau) f_2(\tau) d\tau := (f_1 * f_2)(t) \leftrightarrow i \tilde{f}_1(z) \tilde{f}_2(z);$$

(iii) if  $P$ ,  $Q$ , and  $R$  are differentiable, and  $R(0) = 0$ , then the equation

$$P(t) + \int_0^t dt_1 \int_0^{t_1} Q(t_1 - t_2)P(t_2)dt_2 = R(t), \quad t \geq 0, \quad (2.38)$$

has a unique differentiable solution

$$P(t) = - \int_0^t T(t - t_1)R'(t_1)dt_1, \quad (2.39)$$

where

$$T \leftrightarrow (z + \tilde{Q})^{-1} \quad (2.40)$$

provided by

$$z + \tilde{Q}(z) \neq 0, \quad \Im z < 0. \quad (2.41)$$

Applying the generalized Fourier transform we prove the lemma, which will be often used in what follows:

**Lemma 2.6** *Consider*

$$v(t) = \int_{-2w}^{2w} e^{it\lambda} \rho_{sc}(\lambda) d\lambda, \quad (2.42)$$

where  $\rho_{sc}$  is the density of the semicircle law

$$\rho_{sc}(\lambda) = (2\pi w^2)^{-1} \left( (4w^2 - \lambda^2)^{1/2} \mathbb{I}_{[-2w^2, 2w^2]} \right)^{1/2}. \quad (2.43)$$

Then unique differentiable solutions of integral equations

$$F_1(t) + w^2 \int_0^t dt_1 \int_0^{t_1} v(t_1 - t_2)F_1(t_2)dt_2 = 1, \quad (2.44)$$

$$\begin{aligned} F_2(t_1, t_2) + w^2 \int_0^{t_1} dt_3 \int_0^{t_3} v(t_3 - t_4)F_2(t_4, t_2)dt_4 \\ = -w^2 \int_0^{t_1} dt_3 \int_0^{t_2} v(t_2 - t_4)v(t_3 + t_4)dt_4, \end{aligned} \quad (2.45)$$

$$\begin{aligned} F_3(t_1, t_2) + 2w^2 \int_0^{t_1} dt_3 \int_0^{t_3} v(t_3 - t_4)F_3(t_4, t_2)dt_4 \\ = -2w^2 t_2 \int_0^{t_1} v(t_2 + t_3)dt_3, \end{aligned} \quad (2.46)$$

are given by

$$F_1(t) = v(t), \quad (2.47)$$

$$F_2(t_1, t_2) = v(t_1 + t_2) - v(t_1)v(t_2), \quad (2.48)$$

$$F_3(t_1, t_2) = \frac{1}{2\pi^2} \int_{-2w}^{2w} \int_{-2w}^{2w} \frac{\Delta e(t_1)\Delta e(t_2)}{(\lambda_1 - \lambda_2)^2} \frac{4w^2 - \lambda_1\lambda_2}{\sqrt{4w^2 - \lambda_1^2}\sqrt{4w^2 - \lambda_2^2}} d\lambda_1 d\lambda_2, \quad (2.49)$$

where we denote

$$\Delta e(t) = e^{it\lambda_2} - e^{it\lambda_1}. \quad (2.50)$$

**Proof.** Note first that in fact the generalized Fourier transform  $\tilde{v}$  of  $v$  is the Stiltjes transform of the semicircle law density (2.43):

$$\tilde{v}(z) = \int_{\mathbb{R}} \frac{\rho_{sc}(\lambda) d\lambda}{\lambda - z}, \quad \Im z \neq 0,$$

so that

$$\begin{aligned} w^2 \tilde{v}(z)^2 + z \tilde{v}(z) + 1 &= 0, \\ \tilde{v}(z) &= (2w^2)^{-1} (\sqrt{z^2 - 4w^2} - z), \end{aligned} \quad (2.51)$$

where  $\sqrt{z^2 - 4w^2}$  is defined by the asymptotic  $\sqrt{z^2 - 4w^2} = z + O(z^{-1})$ ,  $z \rightarrow \infty$ . Denote  $\tilde{F}_1(z)$ ,  $\tilde{F}_j(z, t_2)$ ,  $j = 2, 3$  the generalized Fourier transforms of  $F_j$ ,  $j = 1, 2, 3$ . We have for  $\tilde{F}_1$ :

$$\tilde{F}_1(z)(1 + w^2 \tilde{v}(z) z^{-1}) = z^{-1},$$

hence,  $\tilde{F}_1(z) = \tilde{v}(z)$  (see (2.51)), and we get (2.47). We also have for  $F_2$  by (2.39) with  $T = -v$  and  $R'(t) = -w^2 \int_0^{t_2} v(t_2 - t_4) v(t + t_4) dt_4$ :

$$F_2(t_1, t_2) = -w^2 \int_0^{t_1} v(t_1 - t_3) dt_3 \int_0^{t_2} v(t_2 - t_4) v(t_3 + t_4) dt_4$$

and after some calculations one can get

$$\begin{aligned} F_2(t_1, t_2) &= \frac{1}{2} \int_{-2w}^{2w} \int_{-2w}^{2w} \Delta e(t_1) \Delta e(t_2) \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2 \\ &= v(t_1 + t_2) - v(t_1) v(t_2), \end{aligned}$$

where  $\Delta e$  is defined in (2.50). Consider now equation (2.46). In this case we have for  $Q$  of (2.38)

$$z + \tilde{Q}(z) = \sqrt{z^2 - 4w^2} \neq 0, \quad \Im z < 0,$$

so that condition (2.41) is fulfilled. This yields for  $T$  of (2.40)

$$T(t) = -\frac{1}{2\pi i} \int_L \frac{e^{izt} dz}{\sqrt{z^2 - 4w^2}} = -\frac{1}{\pi} \int_{-2w}^{2w} \frac{e^{i\lambda t} d\lambda}{\sqrt{4w^2 - \lambda^2}} \quad (2.52)$$

(we replaced the integral over  $L$  by the integral over the edges of the cut  $[-2w, 2w]$ ). This and (2.39) lead to

$$\begin{aligned} F_3(t_1, t_2) &= \frac{2it_2}{2\pi^2} \int_{-2w}^{2w} e^{it_2\lambda_2} \sqrt{4w^2 - \lambda_2^2} d\lambda_2 \int_{-2w}^{2w} \frac{e^{it_1\lambda_2} - e^{it_1\lambda_1}}{\sqrt{4w^2 - \lambda_1^2}(\lambda_2 - \lambda_1)} d\lambda_1 \\ &= \frac{1}{\pi^2} \int_{-2w}^{2w} \frac{e^{it_1\lambda_1}}{\sqrt{4w^2 - \lambda_1^2}} d\lambda_1 \int_{-2w}^{2w} \frac{\partial}{\partial \lambda_2} (e^{it_2\lambda_2} - e^{it_2\lambda_1}) \frac{\sqrt{4w^2 - \lambda_2^2}}{\lambda_1 - \lambda_2} d\lambda_2, \end{aligned}$$

where we used

$$\int_{-2w}^{2w} \frac{d\lambda_1}{\sqrt{4w^2 - \lambda_1^2}(\lambda_2 - \lambda_1)} = 0, \quad |\lambda_2| \leq 2w.$$

Integrating by parts with respect to  $\lambda_2$ , and writing then the half-sum of the obtained expression and the expression with interchanged variables  $\lambda_1 \longleftrightarrow \lambda_2$ , we get (2.49). ■



### 3 The GOE case

Denote by

$$F[\varphi](t) = \frac{1}{2\pi} \int e^{-it\lambda} \varphi(\lambda) d\lambda \quad (3.1)$$

the standard Fourier transform of  $\varphi$ . Writing the Fourier inversion formula

$$\varphi(\lambda) = \int e^{i\lambda t} F[\varphi](t) dt \quad (3.2)$$

and using the spectral theorem for symmetric matrices, we obtain

$$\xi_n^A[\varphi] = \int \xi_n^A(t) F[\varphi](t) dt, \quad (3.3)$$

where  $\xi_n^A(t)$  is a particular case of  $\xi_n^A[\varphi]$  corresponding to  $\varphi(\lambda) = e^{it\lambda}$ :

$$\begin{aligned} \xi_n^A(t) &= \text{Tr} A^{(n)} U(t), \\ U(t) &= U^{(n)}(t) := e^{itM^{(n)}} \end{aligned} \quad (3.4)$$

(see also (2.18)). Denote

$$v_n(t) = n^{-1} \xi_n^I(t) = n^{-1} \text{Tr} U(t). \quad (3.5)$$

Since for any bounded continuous  $\varphi$

$$\lim_{n \rightarrow \infty} n^{-1} \mathbf{E} \{ \text{Tr} \varphi(M^{(n)}) \} = \int_{-2w}^{2w} \varphi(\lambda) \rho_{sc}(\lambda) d\lambda,$$

where  $M^{(n)}$  is Wigner matrix and  $\rho_{sc}$  is the density of the semicircle law (2.43) (see e.g. [32] and references therein), then we have

$$\lim_{n \rightarrow \infty} \mathbf{E} \{ v_n(t) \} = v(t), \quad (3.6)$$

where  $v$  is defined in (2.42).

In this chapter we consider  $\xi_n^A[\varphi]$  corresponding to the GOE matrix  $M^{(n)} = \widehat{M}^{(n)}$ . In view of the orthogonal invariance of GOE probability measure we have

$$\mathbf{E} \{ U_{jk}(t) \} = \delta_{jk} \mathbf{E} \{ v_n(t) \}, \quad (3.7)$$

so that

$$n^{-1} \mathbf{E} \{ \xi_n^A(t) \} = \mathbf{E} \{ v_n(t) \} n^{-1} \text{Tr} A^{(n)}$$

and

$$\lim_{n \rightarrow \infty} n^{-1} \mathbf{E} \{ \xi_n^A(t) \} = T_A \cdot v(t), \quad (3.8)$$

where  $T_A$  is defined in (1.3). We also have:

**Lemma 3.1** *Let  $\widehat{M}^{(n)}$  be the GOE matrix (2.3). Denote*

$$\xi_n^{A\circ}[\varphi] = \xi_n^A[\varphi] - \mathbf{E} \{ \xi_n^A[\varphi] \}. \quad (3.9)$$

Then for any test-function  $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ , whose Fourier transform (3.1) satisfies the condition

$$\int (1 + |t|) |F[\varphi](t)| dt < \infty, \quad (3.10)$$

we have the bound

$$\mathbf{Var}\{\xi_n^A[\varphi]\} := \mathbf{E}\{|\xi_n^{A\circ}[\varphi]|^2\} \leq c \left( \int (1 + |t|) |F[\varphi](t)| dt \right)^2. \quad (3.11)$$

**Proof.** It follows from Poincaré inequality (2.5) and (2.24) that

$$\begin{aligned} \mathbf{Var}\{\xi_n^A(t)\} &\leq \frac{w^2}{n} \sum_{1 \leq l \leq m \leq n} \beta_{lm}^{-1} \mathbf{E}\{|D_{lm}\xi_n^A(t)|^2\} \\ &\leq \frac{2w^2}{n} \sum_{l,m=1}^n \mathbf{E}\{|(U * A^{(n)}U)_{lm}(t)|^2\} = \frac{2w^2|t|^2}{n} \text{Tr } AA^{(n)T}, \end{aligned}$$

so that

$$\mathbf{Var}\{\xi_n^A(t)\} \leq 2C_A w^2 |t|^2, \quad (3.12)$$

where  $C_A$  is defined in (2.28). By (3.3) and the Schwarz inequality

$$\mathbf{Var}\{\xi_n^A[\varphi]\} \leq \left( \int \mathbf{Var}^{1/2}\{\xi_n^A(t)\} |F[\varphi](t)| dt \right)^2. \quad (3.13)$$

This, (3.10), and (3.12) yield (4.4). ■

In this chapter we find limiting covariance for  $\xi_n^A[\varphi]$  and prove that  $\xi_n^A[\varphi]$  in GOE case satisfies CLT. We have two theorems:

**Theorem 3.2** *Let  $\widehat{M}^{(n)}$  be the GOE matrix (2.3), and  $\varphi_{1,2} : \mathbb{R} \rightarrow \mathbb{R}$  be test functions satisfying (3.10). Denote*

$$\mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\} = \mathbf{E}\{\xi_n^{A\circ}[\varphi_1] \xi_n^A[\varphi_2]\}.$$

Then we have

$$\begin{aligned} C_{GOE}[\varphi_1, \varphi_2] &:= \lim_{n \rightarrow \infty} \mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\} \\ &= \frac{T_A^2}{2\pi^2} \int_{-2w}^{2w} \int_{-2w}^{2w} \frac{\Delta\varphi_1}{\Delta\lambda} \frac{\Delta\varphi_2}{\Delta\lambda} \frac{4w^2 - \lambda_1\lambda_2}{\sqrt{4w^2 - \lambda_1^2} \sqrt{4w^2 - \lambda_2^2}} d\lambda_1 d\lambda_2 \\ &\quad + (T_{A(A+A^T)}/2 - T_A^2) \int_{-2w}^{2w} \int_{-2w}^{2w} \Delta\varphi_1 \Delta\varphi_2 \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2, \end{aligned} \quad (3.14)$$

where  $T_A$  is defined in (1.3),

$$\Delta\varphi = \varphi(\lambda_1) - \varphi(\lambda_2), \quad \Delta\lambda = \lambda_1 - \lambda_2, \quad (3.15)$$

and  $\rho_{sc}$  is the density of the semicircle law (2.43).

**Theorem 3.3** Let  $\widehat{M}^{(n)}$  be the GOE matrix (2.3), and  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  satisfies (3.10). Then the random variable  $\xi_n^{A_0}[\varphi]$  converges in distribution to the Gaussian random variable with zero mean and the variance given by

$$V_{GOE}[\varphi] = \frac{T_A^2}{2\pi^2} \int_{-2w}^{2w} \int_{-2w}^{2w} \left( \frac{\Delta\varphi}{\Delta\lambda} \right)^2 \frac{4w^2 - \lambda_1\lambda_2}{\sqrt{4w^2 - \lambda_1^2} \sqrt{4w^2 - \lambda_2^2}} d\lambda_1 d\lambda_2 \\ + (T_{A(A+A^T)}/2 - T_A^2) \int_{-2w}^{2w} \int_{-2w}^{2w} (\Delta\varphi)^2 \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2. \quad (3.16)$$

**Remark 3.4** Note that  $V_{GOE}[\varphi]$  can be written in the form

$$V_{GOE}[\varphi] = T_A^2 \cdot V_{GOE}^{\mathcal{N}_n}[\varphi] + (T_{A(A+A^T)}/2 - T_A^2) \cdot V_{GOE}^{jj}[\varphi], \quad (3.17)$$

where

$$V_{GOE}^{\mathcal{N}}[\varphi] = \frac{1}{2\pi^2} \int_{-2w}^{2w} \int_{-2w}^{2w} \left( \frac{\Delta\varphi}{\Delta\lambda} \right)^2 \frac{4w^2 - \lambda_1\lambda_2}{\sqrt{4w^2 - \lambda_1^2} \sqrt{4w^2 - \lambda_2^2}} d\lambda_1 d\lambda_2 \quad (3.18)$$

and

$$V_{GOE}^{jj}[\varphi] = \int_{-2w}^{2w} \int_{-2w}^{2w} (\Delta\varphi)^2 \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2 \quad (3.19)$$

are the limiting variances corresponding to the linear eigenvalue statistics (1.4) and matrix elements (1.5), respectively (compare with the results of [24] and [25]).

Besides, we have for limiting variance  $V_{GOE}^{(M\eta,\eta)}[\varphi]$ , corresponding to the bilinear form (1.7):

$$V_{GOE}^{(M\eta,\eta)}[\varphi] = V_{GOE}^{jj}[\varphi] = \int_{-2w}^{2w} \int_{-2w}^{2w} (\Delta\varphi)^2 \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2. \quad (3.20)$$

**Proof. Theorem 3.2.** Since  $\mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\}$  is linear in  $\varphi_{1,2}$ , it suffices to consider real valued  $\varphi_{1,2}$ . Writing the Fourier inversion formula (3.2) and using the linearity of  $\mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\}$  in  $\varphi_{1,2}$  and the spectral theorem for symmetric matrices, we obtain

$$\mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\} = \int \int \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\} F[\varphi_1](t_1) F[\varphi_2](t_2) dt_1 dt_2 \quad (3.21)$$

with  $\xi_n^A(t)$  of (3.4). Similar to (3.12) with the help of Poincaré inequality (2.5) it can be shown that

$$\mathbf{Var}\{\xi_n^{A'}(t)\} \leq ct^2,$$

where  $\xi_n^{A'}(t) = i\mathrm{Tr} A^{(n)} \widehat{M} e^{it_1 \widehat{M}}$ . This, (3.12), and the Schwarz inequality imply the bounds

$$|\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\}| \leq c|t_1||t_2|, \quad (3.22)$$

$$|\partial \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\} / \partial t_i| \leq c|t_1||t_2|, \quad i = 1, 2. \quad (3.23)$$

Hence, in view of (3.10) the integrand in (3.21) admits an integrable and  $n$ -independent upper bound, and by dominated convergence theorem it suffices to prove the pointwise in  $t_{1,2}$  convergence of  $\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\}$  to a certain limit as  $n \rightarrow \infty$ , implying (3.14). It also follows

from (3.22) – (3.23) that there exists a convergent subsequence  $\{\mathbf{Cov}\{\xi_{n_j}^A(t_1), \xi_{n_j}^A(t_2)\}\}_{j=1}^\infty$ . We will show that every such a subsequence has the same limit leading through (3.21) to (3.14).

We can confine ourselves to  $t_{1,2} \geq 0$ , because  $\mathbf{Cov}\{\xi_n^A(-t_1), \xi_n^A(t_2)\} = \overline{\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\}}$ . Consider

$$\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^B(t_2)\} = \mathbf{E}\{\xi_n^A(t_1)\xi_n^{B^\circ}(t_2)\}, \quad (3.24)$$

putting in appropriate moment  $A^{(n)} = B^{(n)}$ . Here  $\xi_n^{A,B}(t_1)$  correspond to  $A^{(n)}, B^{(n)}$  satisfying (1.2) – (1.3) (see (3.4)). By using Duhamel formula (2.17) we can write

$$\begin{aligned} \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^B(t_2)\} &= \mathbf{E}\{\text{Tr} A^{(n)} U(t_1) \xi_n^{B^\circ}(t_2)\} \\ &= i \int_0^{t_1} \sum_{l,m=1}^n \mathbf{E}\{\widehat{M}_{lm}(A^{(n)} U)_{lm}(t_3) \xi_n^{B^\circ}(t_2)\} dt_3. \end{aligned}$$

Applying differentiation formula (2.4) with (2.2) written in the form

$$\mathbf{E}\{(W_{lm}^{(n)})^2\} = w^2 \beta_{lm}^{-1} \quad (3.25)$$

(see (2.21)), and then (2.23) – (2.24), we obtain:

$$\begin{aligned} \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^B(t_2)\} &= iw^2 \int_0^{t_1} \frac{1}{n} \sum_{l,m=1}^n \beta_{lm}^{-1} \mathbf{E}\{D_{lm}[(A^{(n)} U)_{lm}(t_3) \xi_n^{B^\circ}(t_2)]\} dt_3 \\ &= -\frac{w^2}{n} \int_0^{t_1} dt_3 \int_0^{t_3} \mathbf{E}\{[\xi_n^I(t_3 - t_4) \xi_n^A(t_4) + \xi_n^A(t_3)] \xi_n^{B^\circ}(t_2)\} dt_4 \\ &\quad - \frac{w^2}{n} \int_0^{t_1} dt_3 \int_0^{t_2} \mathbf{E}\{\text{Tr} A^{(n)} U(t_3 + t_4) (B^{(n)} + B^{(n)T}) U(t_2 - t_4)\} dt_4. \end{aligned} \quad (3.26)$$

Putting

$$v_n = v_n^\circ + \bar{v}_n, \quad \bar{v}_n = \mathbf{E}\{v_n\}, \quad (3.27)$$

$$\xi_n^A = \xi_n^{A^\circ} + \bar{\xi}_n^A, \quad \bar{\xi}_n^A = \mathbf{E}\{\xi_n^A\}, \quad (3.28)$$

we get from (3.26)

$$\begin{aligned} \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^B(t_2)\} &= -w^2 \int_0^{t_1} dt_3 \int_0^{t_3} \bar{v}_n(t_3 - t_4) \mathbf{E}\{\xi_n^A(t_4) \xi_n^{B^\circ}(t_2)\} dt_4 \\ &\quad - \frac{w^2}{n} \int_0^{t_1} dt_3 \int_0^{t_3} \bar{\xi}_n^A(t_3 - t_4) \mathbf{E}\{\xi_n^I(t_4) \xi_n^{B^\circ}(t_2)\} dt_4 \\ &\quad - \frac{w^2}{n} \int_0^{t_1} dt_3 \int_0^{t_2} \mathbf{E}\{\text{Tr} A^{(n)} U(t_3 + t_4) (B^{(n)} + B^{(n)T}) U(t_2 - t_4)\} dt_4 \\ &\quad + r_n(t_1, t_2), \end{aligned} \quad (3.29)$$

where

$$r_n(t_1, t_2) = -w^2 \int_0^{t_1} \mathbf{E}\{[(v_n^\circ * \xi_n^{A^\circ})(t_3) + t_3 n^{-1} \xi_n^A(t_3)] \xi_n^{B^\circ}(t_2)\} dt_3. \quad (3.30)$$

With the help of Poincaré inequality (2.5) it can be shown that

$$\mathbf{Var}\{v_n^\circ \xi_n^{A^\circ}\} = O(n^{-2}), \quad n \rightarrow \infty,$$

which together with (3.12) yield

$$r_n(t_1, t_2) = O(n^{-1}), \quad n \rightarrow \infty. \quad (3.31)$$

Consider convergent subsequence  $\{\mathbf{Cov}\{\xi_{n_j}^A(t_1), \xi_{n_j}^B(t_2)\}\}_{j=1}^\infty$  and denote

$$C^{A,B}(t_1, t_2) := \lim_{n_j \rightarrow \infty} \mathbf{Cov}\{\xi_{n_j}^A(t_1), \xi_{n_j}^B(t_2)\}.$$

It follows from (3.6), (3.8), and (3.29) – (3.31) that  $C^{A,B}(t_1, t_2)$  satisfies the equation

$$\begin{aligned} C^{A,B}(t_1, t_2) &+ w^2 \int_0^{t_1} (v * C^{A,B}(\cdot, t_2))(t_3) dt_3 \\ &= -w^2 T_A \int_0^{t_1} (v * C^{I,B}(\cdot, t_2))(t_3) dt_3 \\ &\quad - w^2 \int_0^{t_1} dt_3 \int_0^{t_2} \lim_{n_j \rightarrow \infty} \mathbf{E}\{n_j^{-1} \text{Tr} A^{(n)} U(t_3 + t_4) (B^{(n)} + B^{(n)T}) U(t_2 - t_4)\} dt_4. \end{aligned} \quad (3.32)$$

In particular, putting in (3.32)  $A^{(n)} = I$  we get

$$C^{I,B}(t_1, t_2) + 2w^2 \int_0^{t_1} (v * C^{I,B}(\cdot, t_2))(t_3) dt_3 = -2w^2 T_B t_2 \int_0^{t_1} v(t_2 + t_3) dt_3, \quad (3.33)$$

so that by (2.49)

$$C^{I,B}(t_1, t_2) = \frac{T_B}{2\pi^2} \int_{-2w}^{2w} \int_{-2w}^{2w} \frac{\Delta e(t_1) \Delta e(t_2)}{(\lambda_1 - \lambda_2)^2} \frac{4w^2 - \lambda_1 \lambda_2}{\sqrt{4w^2 - \lambda_1^2} \sqrt{4w^2 - \lambda_2^2}} d\lambda_1 d\lambda_2. \quad (3.34)$$

Now let us calculate the second term in the r.h.s. of (3.32). Consider  $\mathbf{E}\{n^{-1} \text{Tr} A^{(n)} U(t_1) C^{(n)} U(t_2)\}$ . We have by (3.7)

$$\mathbf{E}\{n^{-1} \text{Tr} A^{(n)} U(t_1) C^{(n)} U(t_2)\} = T_{AC} \bar{v}_n(t_1) \bar{v}_n(t_2) + F_n(t_1, t_2), \quad (3.35)$$

where  $\bar{v}_n$  is defined in (3.27) and

$$F_n(t_1, t_2) = n^{-1} \sum_{j,l=1}^n \mathbf{E}\{(U C^{(n)})_{jl}(t_1) (U^\circ A^{(n)})_{lj}(t_2)\}.$$

Repeating steps leading from (3.24) to (3.32) and using consequently Duhamel formula (2.17) and the differentiation formulas (2.4) and (2.23) – (2.24), one can easily get

$$\begin{aligned} F_n(t_1, t_2) &+ w^2 \int_0^{t_1} (\bar{v}_n * F_n(\cdot, t_2))(t_3) dt_3 \\ &= -w^2 \int_0^{t_1} dt_3 \int_0^{t_2} n^{-1} \bar{\xi}_n^A(t_3 + t_4) \cdot n^{-1} \bar{\xi}_n^C(t_2 - t_4) dt_4 + r_n^1(t_1, t_2), \end{aligned} \quad (3.36)$$

with  $\bar{\xi}_n^A$  of (3.28), and

$$\begin{aligned} r_n^1(t_1, t_2) &= -w^2 \int_0^{t_1} dt_3 \mathbf{E}\left\{ \left[ (v_n^\circ * F_n(\cdot, t_2))(t_3) + t_3 n^{-1} F_n(t_3, t_2) \right. \right. \\ &\quad \left. \left. + n^{-2} \int_0^{t_2} \text{Tr} A^{(n)T} U(t_3 + t_4) C^{(n)} U(t_2 - t_4) + \xi_n^{A^\circ}(t_3 + t_4) \xi_n^{C^\circ}(t_2 - t_4) \right] dt_4 \right\}. \end{aligned}$$

It follows from (3.12) that

$$r_n^1(t_1, t_2) = O(n^{-1}), \quad n \rightarrow \infty.$$

This, (3.6), (3.8), and (3.36), yield for  $F = \lim_{n_j \rightarrow \infty} F_{n_j}$ :

$$\begin{aligned} F(t_1, t_2) + w^2 \int_0^{t_1} (v * F(\cdot, t_2))(t_3) dt_3 \\ = -w^2 T_A T_C \int_0^{t_1} dt_3 \int_0^{t_2} v(t_3 + t_4) v(t_2 - t_4) dt_4. \end{aligned} \quad (3.37)$$

Hence,  $F(t_1, t_2) = T_A T_C F_1(t_1, t_2)$  with  $F_1$  of (2.47). This, (3.35), and (3.6) yield

$$\begin{aligned} F(t_1, t_2) &= -w^2 T_A T_C \int_0^{t_1} dt_3 \int_0^{t_2} v(t_1 - t_3) v(t_3 + t_4) v(t_2 - t_4) dt_4 \\ &= \frac{T_A T_C}{2} \int_{-2w}^{2w} \int_{-2w}^{2w} \Delta e(t_1) \Delta e(t_2) \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2 \\ &= T_A T_C (v(t_1 + t_2) - v(t_1) v(t_2)), \end{aligned}$$

where  $\Delta e$  is defined in (2.50). This, (2.42), and (3.35) leads to

$$\lim_{n_j \rightarrow \infty} \mathbf{E}\{n^{-1} \text{Tr} A^{(n)} U(t_1) C^{(n)} U(t_2)\} = T_A T_C v(t_1 + t_2) + (T_{AC} - T_A T_C) v(t_1) v(t_2). \quad (3.38)$$

Putting (3.34) and (3.38) with  $C^{(n)} = B^{(n)} + B^{(n)T}$  in (3.32) we obtain the equation for  $C^{A,B}$

$$\begin{aligned} C^{A,B}(t_1, t_2) + w^2 \int_0^{t_1} (v * C^{A,B}(\cdot, t_2))(t_3) dt_3 \\ = -w^2 \int_0^{t_1} \left[ T_A (v * C^{I,B}(\cdot, t_2))(t_3) + 2T_A T_B t_2 v(t_3 + t_2) \right. \\ \left. (T_{A(B+B^T)} - 2T_A T_B) \int_0^{t_2} v(t_3 + t_4) v(t_2 - t_4) dt_4 \right] dt_3, \end{aligned}$$

solving which with the help of Lemma 2.6 we finally get

$$\begin{aligned} C^{A,B}(t_1, t_2) &= \frac{T_A T_B}{2\pi^2} \int_{-2w}^{2w} \int_{-2w}^{2w} \frac{\Delta e(t_1) \Delta e(t_2)}{(\lambda_1 - \lambda_2)^2} \frac{4w^2 - \lambda_1 \lambda_2}{\sqrt{4w^2 - \lambda_1^2} \sqrt{4w^2 - \lambda_2^2}} d\lambda_1 d\lambda_2 \\ &\quad + (T_{A(B+B^T)}/2 - T_A T_B) \int_{-2w}^{2w} \int_{-2w}^{2w} \Delta e(t_1) \Delta e(t_2) \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2. \end{aligned}$$

Putting this expression with  $A^{(n)} = B^{(n)}$  in (3.21) we obtain (3.14) and so prove the theorem.  $\blacksquare$

**Proof. Theorem 3.3.** The detailed proofs of CLTs for linear eigenvalue statistics (1.4) and for matrix elements (1.5) are given in [23, 24] and [25], respectively. The proof of Theorem 3.3 follows the same scheme, so here we only outline its main steps.

By the continuity theorem for characteristic functions it suffices to show that if

$$Z_n^A(x) = \mathbf{E}\{e^{ix\xi_n^{A \circ}[\varphi]}\}, \quad (3.39)$$

then for any  $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} Z_n^A(x) = Z^A(x), \quad (3.40)$$

where

$$Z^A(x) = \exp \left\{ -x^2 V_{GOE}[\varphi]/2 \right\}. \quad (3.41)$$

We obtain (3.41), hence the theorem, for a class of test functions satisfying condition

$$\int (1 + |t|^2) |F[\varphi](t)| dt < \infty. \quad (3.42)$$

(cf (3.10)), then the theorem can be extended to the class of functions satisfying (3.10) by using a standard approximation procedure.

Since  $Z_n^A(0) = 1$  and  $Z_n^A(x)$  is continuous, we can write the relation

$$Z_n^A(x) = 1 + \int_0^x Z_n^{A'}(y) dy, \quad x \in \mathbb{R}, \quad (3.43)$$

showing that it suffices to prove that the sequence  $\{Z_n^{A'}\}$  is uniformly bounded on any finite interval and that for any converging subsequences  $\{Z_{n_i}^A\}_{i \geq 1}$  and  $\{Z_{n_i}^{A'}\}_{i \geq 1}$  there exists  $Z^A(x)$ , such that

$$\lim_{i \rightarrow \infty} Z_{n_i}^A(x) = Z^A(x), \quad (3.44)$$

and

$$\lim_{i \rightarrow \infty} Z_{n_i}^{A'}(x) = -x V_{GOE}[\varphi] Z^A(x). \quad (3.45)$$

Indeed, if yes, then  $Z^A(x)$  is a continuous function satisfying equation

$$Z^A(x) = 1 - V_{GOE}[\varphi] \int_0^x y Z^A(y) dy, \quad x \in \mathbb{R}, \quad (3.46)$$

which is uniquely soluble in the class of bounded continuous functions, and its solution is evidently (3.41).

We denote

$$e_n(x) = \exp\{ix \xi_n^{A\circ}[\varphi]\}, \quad (3.47)$$

and write according to (3.2) and (3.39)

$$Z_n^{A'}(x) = i \mathbf{E} \left\{ \xi_n^{A\circ}[\varphi] e^{ix \xi_n^{A\circ}[\varphi]} \right\} = i \int F[\varphi](t) Y_n^A(x, t) dt, \quad (3.48)$$

where

$$Y_n^A(x, t) = \mathbf{E} \left\{ \xi_n^A(t) e_n^\circ(x) \right\}, \quad (3.49)$$

and  $\xi_n^A(t)$  is defined in (3.4). It follows from the Schwarz inequality and (3.12) that

$$|Y_n^A(x, t)| \leq c|t|.$$

This and (3.42) yield that the sequence  $Z_n^{A'}$  is uniformly bounded. Hence, there is a convergent subsequence  $Z_{n_i}^A$ , and by the dominated convergence theorem to find its limit as  $n \rightarrow \infty$  it suffices to find the pointwise limit of the corresponding subsequence  $Y_{n_i}^A$ . It also can be shown with the help of Poincaré inequality (2.5) and (3.42) that sequences  $\{\partial Y_n^A / \partial x\}$  and

$\{\partial Y_n^A/\partial t\}$  are uniformly bounded in  $(t, x) \in K \subset \mathbb{R}_+^2$ ,  $n \in \mathbb{N}$ , for any bounded  $K$ , so that the sequence  $\{Y_n^A\}$  is equicontinuous on any finite set of  $\mathbb{R}_+^2$ , and contains convergent subsequences. Hence, for any converging subsequence  $\{Z_{n_i}^A\}$  (see (3.44)) there is a converging subsequence  $\{Y_{n_i'}^A\}$  and continuous function  $Y^A$  (which obviously depends on  $\{Z_{n_i}^A\}$ ) such that

$$\lim_{n_i' \rightarrow \infty} Y_{n_i'}^A = Y^A, \quad \lim_{n_i' \rightarrow \infty} Z_{n_i'}^A = Z^A. \quad (3.50)$$

We will show now that  $Y^A$  satisfies certain integral equation leading through (3.48) to (3.46), hence, to (3.41). This will finish the proof of the theorem under condition (3.42).

Applying consequently the Duhamel formula (2.17) and differentiation formula (2.4) with (3.25), we get

$$\begin{aligned} Y_n^A(x, t) &= \frac{i}{\sqrt{n}} \int_0^t \sum_{j,k=1}^n \mathbf{E}\{\widehat{W}_{jk}(UA^{(n)})_{kj}(t_1)e_n^\circ(x)\} dt_1 \\ &= \frac{iw^2}{n} \int_0^t \sum_{j,k=1}^n \beta_{jk}^{-1} \mathbf{E}\{D_{jk}((UA^{(n)})_{kj}(t_1)e_n^\circ(x))\} dt_1, \end{aligned}$$

where  $D_{jk} = \partial/\partial M_{jk}$ . It follows from (2.24) that

$$D_{jk}e_n(x) = -\beta_{jk}xe_n(x) \int (U * C^{(n)}U)_{jk}(\theta)F[\varphi](\theta)d\theta. \quad (3.51)$$

This, (2.23), (3.27) – (3.28), and relation  $e_n = e_n^\circ + Z_n^A$  yield

$$\begin{aligned} Y_n^A(x, t) &= -w^2 \int_0^t [\bar{v}_n * Y_n^A(x, \cdot) + \bar{\xi}_n * Y_n^I(x, \cdot)](t_1) dt_1 \\ &\quad - iw^2 x Z_n^A(x) \int_0^t dt_1 \int F[\varphi](\theta) d\theta \int_0^\theta n^{-1} \mathbf{E}\{\text{Tr} A^{(n)} U(\theta - \theta_1) C^{(n)} U(\theta_1 + t_1)\} d\theta_1 \\ &\quad + r_n(x, t), \end{aligned}$$

where

$$\begin{aligned} r_n(x, t) &= -w^2 \int_0^t \mathbf{E}\{[(v_n^\circ * \xi_n^{A^\circ})(t_1) + t_1 n^{-1} \xi_n^A(t_1)]e_n^\circ(x)\} dt_1 \\ &\quad - iw^2 x \int_0^t dt_1 \int F[\varphi](\theta) d\theta \int_0^\theta n^{-1} \mathbf{E}\{\text{Tr} A^{(n)} U(\theta - \theta_1) C^{(n)} U(\theta_1 + t_1) e_n^\circ(x)\} d\theta_1. \end{aligned}$$

With the help of Poincaré inequality (2.5) it can be shown that

$$\mathbf{Var}\{n^{-1} \text{Tr} A^{(n)} U(\theta - \theta_1) C^{(n)} U(\theta_1 + t_1)\} \leq c(|\theta|^2 + |\theta_1|^2 + |t|^2)n^{-2},$$

which together with (3.12) and (3.42) yield

$$r_n(t_1, t_2) = O(n^{-1}), \quad n \rightarrow \infty. \quad (3.52)$$

This and (3.6) leads to equation with respect to  $Y^A = \lim_{n_j \rightarrow \infty} Y_{n_j}^A$ :

$$\begin{aligned} Y^A(x, t) &+ w^2 \int_0^t (v * Y^A(x, \cdot))(t_1) dt_1 \\ &= -w^2 T_A \int_0^t (v * Y^I(x, \cdot))(t_1) dt_1 \\ &\quad - iw^2 x Z^A(x) \int_0^t dt_1 \int F[\varphi](\theta) d\theta \int_0^\theta \lim_{n_j \rightarrow \infty} \mathbf{E}\{n^{-1} \text{Tr} A^{(n)} U(\theta - \theta_1) C^{(n)} U(\theta_1 + t_1)\} d\theta_1, \end{aligned} \quad (3.53)$$



where  $Y^I$  is a solution of the equation

$$\begin{aligned} Y^I(x, t) + 2w^2 \int_0^t (v * Y^I(x, \cdot))(t_1) dt_1 \\ = -2iw^2 x Z^A(x) T_A \int_0^t dt_1 \int F[\varphi](\theta) \theta v(\theta + t_1) d\theta. \end{aligned} \quad (3.54)$$

Comparing pairs of equations (3.32) – (3.33) and (3.53) – (3.54) one can see that

$$Y^A(x, t) = -ix Z^A(x) \int C^{A,A}(t, \theta) F[\varphi](\theta) d\theta,$$

where  $C^{A,A}$  is given by (3.24) with  $A = B$ . This and (3.48) yield

$$Z^A(x) = x Z^A(x) \int \int C^{A,A}(t, \theta) F[\varphi](t) F[\varphi](\theta) dt d\theta = x Z^A(x) V_{GOE}[\varphi]$$

(see (3.21), (3.24), and (3.16)), and so leads to (3.46) and completes the proof of the theorem.  $\blacksquare$

## 4 Covariance for $\xi_n^A[\varphi]$ in the Wigner case

We show first that if  $M^{(n)}$  is the Wigner matrix with uniformly bounded eighth moments of its entries, and the test-function  $\varphi$  is essentially of class  $\mathbf{C}^4$ , then the variance of  $\xi_n^A[\varphi]$  is of the order  $O(1)$  as  $n \rightarrow \infty$ . We have

**Lemma 4.1** *Let  $M^{(n)} = n^{-1/2} W^{(n)}$  be the real symmetric Wigner matrix (2.1) – (2.2). Assume that:*

(i) *the third moments of its entries do not depend on  $j, k$ , and  $n$ :*

$$\mu_3 = \mathbf{E}\{(W_{jk}^{(n)})^3\}; \quad (4.1)$$

(ii) *the eighth moments are uniformly bounded:*

$$w_8 := \sup_{n \in \mathbb{N}} \max_{1 \leq j, k \leq n} \mathbf{E}\{(W_{jk}^{(n)})^8\} < \infty. \quad (4.2)$$

Then for any test-function  $\varphi : \mathbb{R} \rightarrow \mathbb{C}$ , whose Fourier transform (3.1) satisfies the condition

$$\int (1 + |t|)^4 |F[\varphi](t)| dt < \infty, \quad (4.3)$$

we have the bound

$$\begin{aligned} \mathbf{Var}\{\xi_n^A[\varphi]\} &= \mathbf{E}\{|\xi_n^{A\circ}[\varphi]|^2\} \\ &\leq c \left( \int (1 + |t|)^4 |F[\varphi](t)| dt \right)^2. \end{aligned} \quad (4.4)$$

The proof of (4.4) follows from (3.13), (4.3), and bound

$$\mathbf{Var}\{\xi_n^A(t)\} \leq c(1 + |t|)^8 \quad (4.5)$$

(see (6.7)).

**Theorem 4.2** Let  $M^{(n)} = n^{-1/2}W^{(n)}$  be the real symmetric Wigner matrix (2.1) – (2.2), whose third and fourth moments do not depend on  $j, k$ , and  $n$ :

$$\mu_3 = \mathbf{E}\{(W_{jk}^{(n)})^3\}, \quad \mu_4 = \mathbf{E}\{(W_{jk}^{(n)})^4\}, \quad (4.6)$$

and the eighth moments are uniformly bounded (see (4.2)). Let  $\{A^{(n)}\}_{n=1}^\infty$  satisfies (1.2) – (1.3),  $C^{(n)} = A^{(n)} + A^{(n)T}$ , and there exist

$$K_A^{(1)} = \lim_{n \rightarrow \infty} n^{-3/2} \sum_{l,m=1}^n A_{ll}^{(n)} C_{lm}^{(n)} \quad (4.7)$$

$$K_A^{(2)} = T_A \lim_{n \rightarrow \infty} n^{-3/2} \sum_{l,m=1}^n C_{lm}^{(n)}, \quad (4.8)$$

$$K_A^{(3)} = \lim_{n \rightarrow \infty} n^{-1} \sum_{m=1}^n A_{mm}^{(n)} \left( A_{mm}^{(n)} - n^{-1} \text{Tr } A^{(n)} \right). \quad (4.9)$$

Then we have for any  $\varphi_{1,2} : \mathbb{R} \rightarrow \mathbb{R}$  satisfying (4.3):

$$\lim_{n \rightarrow \infty} \mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\} = C_{GOE}[\varphi_1, \varphi_2] + C_{\kappa_3}[\varphi_1, \varphi_2] + C_{\kappa_4}[\varphi_1, \varphi_2], \quad (4.10)$$

where  $C_{GOE}[\varphi_1, \varphi_2]$  is defined in (3.14),

$$\begin{aligned} C_{\kappa_3}[\varphi_1, \varphi_2] &= \frac{\kappa_3}{w^6} \int_{-2w}^{2w} \int_{-2w}^{2w} \lambda_1 \left( K_A^{(1)} (\lambda_2^2 - w^2) + K_A^{(2)} \left( \frac{2w^4}{4w^2 - \lambda_2^2} - \lambda_2^2 \right) \right) \\ &\quad \times (\varphi_1(\lambda_1) \varphi_2(\lambda_2) + \varphi_1(\lambda_2) \varphi_2(\lambda_1)) \prod_{j=1}^2 \rho_{sc}(\lambda_j) d\lambda_j, \end{aligned} \quad (4.11)$$

$$\begin{aligned} C_{\kappa_4}[\varphi_1, \varphi_2] &= \frac{\kappa_4}{w^8} \left[ K_A^{(3)} \prod_{j=1}^2 \int_{-2w}^{2w} \varphi_j(\lambda) (w^2 - \lambda^2) \rho_{sc}(\lambda) d\lambda \right. \\ &\quad \left. + \frac{T_A^2}{2\pi^2} \prod_{j=1}^2 \int_{-2w}^{2w} \varphi_j(\lambda) \frac{2w^2 - \lambda^2}{\sqrt{4w^2 - \lambda^2}} d\lambda \right], \end{aligned} \quad (4.12)$$

$\kappa_3 = \mu_3$ , and

$$\kappa_4 = \mu_4 - 3w^4 \quad (4.13)$$

is the fourth cumulant of the off-diagonal entries (see (2.7)). In particular,

$$V_W[\varphi] := \lim_{n \rightarrow \infty} \mathbf{Var}\{\xi_n^A[\varphi]\} = V_{GOE}[\varphi] + C_{\kappa_3}[\varphi, \varphi] + C_{\kappa_4}[\varphi, \varphi] \quad (4.14)$$

with  $V_{GOE}[\varphi]$  of (3.16).

**Remark 4.3** Note that for the limiting variances  $V_W^{\mathcal{N}}[\varphi]$  and  $V_W^{jj}[\varphi]$  of linear eigenvalue statistics (1.4) and matrix elements (1.5) we get, respectively:

$$V_W^{\mathcal{N}}[\varphi] = V_{GOE}^{\mathcal{N}}[\varphi] + \frac{\kappa_4}{2\pi^2 w^8} \left| \int_{-2w}^{2w} \varphi(\lambda) \frac{2w^2 - \lambda^2}{\sqrt{4w^2 - \lambda^2}} d\lambda \right|^2 \quad (4.15)$$

and

$$V_W^{jj}[\varphi] = V_{GOE}^{jj}[\varphi] + \frac{\kappa_4}{w^8} \left| \int_{-2w}^{2w} \varphi(\lambda)(w^2 - \lambda^2) \rho_{sc}(\lambda) d\lambda \right|^2, \quad (4.16)$$

where  $V_{GOE}^{\mathcal{N}}[\varphi]$  and  $V_{GOE}^{jj}[\varphi]$  are defined in (3.18) – (3.19). This coincides with the results of [24] and [26].

**Remark 4.4** In case of bilinear forms (see (1.6) – (1.7))  $T_A = 0$  and coefficients  $K_A^{(j)}$ ,  $j = 1, 2, 3$  of (4.7) – (4.9) take form

$$\begin{aligned} K_A^{(1)} &= 2 \lim_{n \rightarrow \infty} n^{-1/2} \sum_{m=1}^n \eta_m^{(n)} \sum_{l=1}^n (\eta_l^{(n)})^3, \\ K_A^{(2)} &= 2T_A \left( \lim_{n \rightarrow \infty} n^{-1/2} \sum_{m=1}^n \eta_m^{(n)} \right)^2 = 0, \\ K_A^{(3)} &= \lim_{n \rightarrow \infty} \sum_{m=1}^n (\eta_m^{(n)})^4. \end{aligned}$$

In particular, if  $\eta_m^{(n)} = O(n^{-1/2})$ ,  $n \rightarrow \infty$  for all  $m = 1, \dots, n$ , then  $K_A^{(j)} = 0$ ,  $j = 1, 2, 3$ , and we get for the limiting variance:

$$V_W^{(M\eta, \eta)}[\varphi] = V_{GOE}^{(M\eta, \eta)}[\varphi] = \int_{-2w}^{2w} \int_{-2w}^{2w} (\Delta\varphi)^2 \rho_{sc}(\lambda_1) \rho_{sc}(\lambda_2) d\lambda_1 d\lambda_2 \quad (4.17)$$

(see (3.20)).

**Remark 4.5** We choose here the Wigner matrix so that its first two moments matches the first two moments of the GOE matrix (see (2.2)). This fact allows to use known properties of GOE and lies at the basis of interpolation procedure widely used in the proof of Lemma 6.1 below. In fact this condition is pure technical one, and we can replace condition (2.2) with more general one and consider Wigner matrix  $\widetilde{M} = n^{-1/2} \widetilde{W}$ , satisfying

$$\begin{aligned} \mathbf{E}\{\widetilde{W}_{jk}^{(n)}\} &= 0, \quad 1 \leq j \leq k \leq n, \\ \mathbf{E}\{(\widetilde{W}_{jk}^{(n)})^2\} &= w^2, \quad j \neq k, \quad \mathbf{E}\{(\widetilde{W}_{jj}^{(n)})^2\} = w_2 w^2, \quad w_2 > 0. \end{aligned} \quad (4.18)$$

In this case there arise additional terms in (4.10) and (4.14) proportional to  $w_2 - 2$ . In particular, we have for the corresponding limiting variance

$$\begin{aligned} V_{\widetilde{W}}^{w_2}[\varphi] &= V_W[\varphi] + (w_2 - 2)w^{-2} \left( K_A^{(3)} \left( \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu \right)^2 \right. \\ &\quad \left. + T_A^2 \left( \frac{1}{2\pi} \int_{-2w}^{2w} \frac{\varphi(\mu) \mu}{\sqrt{4w^2 - \mu^2}} d\mu \right)^2 \right), \end{aligned} \quad (4.19)$$

where  $V_W[\varphi]$  is given by (4.14).

**Proof.** We write as in the GOE case (see (3.21)):

$$\mathbf{Cov}\{\xi_n^A[\varphi_1], \xi_n^A[\varphi_2]\} = \int \int \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\} \prod_{j=1}^2 F[\varphi_j](t_j) dt_j, \quad (4.20)$$

and note that in view of (4.5) and (4.3) the integrand admits an integrable and  $n$ -independent upper bound. By dominated convergence theorem it suffices to prove the pointwise in  $t_{1,2}$  convergence of  $\mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\}$  to a certain limit as  $n \rightarrow \infty$ , implying (4.10). To do this we use known result for the GOE matrix (see Theorem 3.2) and an interpolating procedure proposed in [21].

Let  $\widehat{M}^{(n)} = n^{-1/2} \widehat{W}^{(n)}$  be the GOE matrix (2.3) independent of  $M^{(n)}$ , and

$$\widehat{U}(t) = \widehat{U}^{(n)}(t) := e^{it\widehat{M}^{(n)}}, \quad \widehat{\xi}_n^A(t) = \text{Tr } A^{(n)} \widehat{U}(t). \quad (4.21)$$

Consider the "interpolating" random matrix

$$M^{(n)}(s) = s^{1/2} M^{(n)} + (1-s)^{1/2} \widehat{M}^{(n)}, \quad 0 \leq s \leq 1, \quad (4.22)$$

viewed as defined on the product of the probability spaces of matrices  $W^{(n)}$  and  $\widehat{W}^{(n)}$  (cf (3.4)). We denote again by  $\mathbf{E}\{\dots\}$  the corresponding expectation in the product space. Since  $M^{(n)}(1) = M^{(n)}$ ,  $M^{(n)}(0) = \widehat{M}^{(n)}$ , then putting

$$U(t, s) = U^{(n)}(t, s) := e^{itM^{(n)}(s)}, \quad \xi_n^A(t, s) = \text{Tr } A^{(n)} U(t, s), \quad (4.23)$$

we can write

$$\begin{aligned} C_n^\Delta(t_1, t_2) &:= \mathbf{Cov}\{\xi_n^A(t_1), \xi_n^A(t_2)\} - \mathbf{Cov}\{\widehat{\xi}_n^A(t_1), \widehat{\xi}_n^A(t_2)\} \\ &= \int_0^1 \frac{\partial}{\partial s} \mathbf{E}\{\xi_n^A(t_1, s) \xi_n^{Ao}(t_2, s)\} ds = c_n^\Delta(t_1, t_2) + c_n^\Delta(t_2, t_1), \end{aligned} \quad (4.24)$$

where

$$\begin{aligned} c_n^\Delta(t_1, t_2) &= \int_0^1 \mathbf{E}\left\{ \frac{\partial}{\partial s} (\xi_n^A(t_1, s)) \cdot \xi_n^{Ao}(t_2, s) \right\} ds \\ &= \frac{i}{2} \int_0^1 \left( \frac{1}{\sqrt{ns}} \sum_{l,m=1}^n \mathbf{E}\{W_{lm}^{(n)} \Phi_{lm}\} - \frac{1}{\sqrt{n(1-s)}} \sum_{l,m=1}^n \mathbf{E}\{\widehat{W}_{lm} \Phi_{lm}\} \right) ds \end{aligned} \quad (4.25)$$

and

$$\Phi_{lm} = \Phi_{lm}(t_1, t_2, s) = (U * A^{(n)} U)_{ml}(t_1, s) \xi_n^{Ao}(t_2, s). \quad (4.26)$$

A simple algebra based on (2.19) – (2.30) allows to obtain

$$|D_{lm}^q \Phi_{lm}| \leq C_q (1 + |t_1| + |t_2|)^{q+1} n^{3/2}, \quad (4.27)$$

with  $C_q$  depending only on  $q \in \mathbb{N}$ . Besides, since

$$\frac{\partial}{\partial W_{lm}^{(n)}} = \sqrt{\frac{s}{n}} D_{lm}(s), \quad D_{lm}(s) = \frac{\partial}{\partial M_{lm}^{(n)}(s)},$$

then every derivative with respect to  $W_{lm}^{(n)}$  gives the factor  $n^{-1/2}$ . Hence, applying differentiation formula (2.8) with  $\zeta = W_{lm}^{(n)}$ ,  $p = 6$ , and  $\Phi = \Phi_{lm}$  to every term of the first sum and differentiation formula (2.4) to every term of the second sum in the r.h.s. of (4.25), we obtain (see also (2.10)):

$$c_n^\Delta(t_1, t_2) = \frac{i}{2} \int_0^1 \left[ \sum_{j=2}^6 s^{(j-1)/2} T_j^{(n)} + \varepsilon_6 \right] ds, \quad (4.28)$$

where

$$T_j^{(n)} = \frac{1}{j!n^{(j+1)/2}} \sum_{l,m=1}^n \kappa_{j+1,lm} \mathbf{E}\{D_{lm}^j \Phi_{lm}\}, \quad j = 2, \dots, 6, \quad (4.29)$$

and by (2.9) and (4.27)

$$|\varepsilon_6| \leq \frac{C_6 w_8}{n^4} \sum_{l,m=1}^n \sup_{M \in \mathcal{S}_n} |D_{lm}^7 \Phi_{lm}| \leq c(1 + |t_1| + |t_2|)^8 n^{-1/2}. \quad (4.30)$$

Now it follows from Lemma 4.6 below that

$$\int \int \left[ \frac{i}{2} \int_0^1 s^{1/2} \lim_{n \rightarrow \infty} (T_2^{(n)}(t_1, t_2) + T_2^{(n)}(t_2, t_1)) ds \right] \prod_{j=1}^2 F[\varphi_j](t_j) dt_j = C_{\kappa_3}[\varphi_1, \varphi_2], \quad (4.31)$$

$$\int \int \left[ \frac{i}{2} \int_0^1 s \lim_{n \rightarrow \infty} (T_3^{(n)}(t_1, t_2) + T_3^{(n)}(t_2, t_1)) ds \right] \prod_{j=1}^2 F[\varphi_j](t_j) dt_j = C_{\kappa_4}[\varphi_1, \varphi_2], \quad (4.32)$$

and

$$\lim_{n \rightarrow \infty} T_j^{(n)} = 0, \quad j = 4, 5, 6, \quad (4.33)$$

with  $C_{\kappa_3}[\varphi_1, \varphi_2]$ ,  $C_{\kappa_4}[\varphi_1, \varphi_2]$  of (4.11) – (4.12). This leads through (4.28), (4.24), and (3.14) to (4.10) – (4.12) and completes the proof. ■

**Lemma 4.6** *Under conditions of Theorem 4.2 the statements (4.31) – (4.33) are valid.*

**Proof.** Consider  $T_2^{(n)}$  of (4.29) and note that by (2.7) and (4.1)  $\kappa_{3,lm} = \mu_3 = \kappa_3$ , and we have

$$\begin{aligned} T_2^{(n)}(t_1, t_2, s) &= \frac{\kappa_3}{2n^{3/2}} \sum_{l,m=1}^n \mathbf{E}\{D_{lm}^2((U * A^{(n)}U)_{ml}(t_1, s) \xi_n^{A^\circ}(t_2, s))\} \\ &= \frac{\kappa_3}{2n^{3/2}} \sum_{l,m=1}^n \mathbf{E}\{\xi_n^{A^\circ}(t_2, s) D_{lm}^2(U * A^{(n)}U)_{ml}(t_1, s) \\ &\quad + 2D_{lm}(U * A^{(n)}U)_{ml}(t_1, s) D_{lm} \xi_n^A(t_2, s) \\ &\quad + (U * A^{(n)}U)_{ml}(t_1, s) D_{lm}^2 \xi_n^A(t_2, s)\} =: \kappa_3 [T_{21}^{(n)} + T_{22}^{(n)} + T_{23}^{(n)}]. \end{aligned} \quad (4.34)$$

Consider  $T_{21}^{(n)}$ . It follows from (2.20) and (2.26) that  $D_{lm}^2(U * A^{(n)}U)_{ml}$  of  $T_{21}^{(n)}$  gives the terms of the form

$$T_{21}^{1(n)} = n^{-3/2} \sum_{l,m=1}^n U_{lm} U_{lm} (U A^{(n)} U)_{lm}, \quad (4.35)$$

$$T_{21}^{2(n)} = n^{-3/2} \sum_{l,m=1}^n U_{lm} U_{ll} (U A^{(n)} U)_{mm}, \quad (4.36)$$

$$T_{21}^{3(n)} = n^{-3/2} \sum_{l,m=1}^n U_{ll} U_{mm} (U A^{(n)} U)_{lm}, \quad (4.37)$$

Here for shortness we omit the sign of conjugation "  $*$  " and arguments of  $U$ . Besides, we replace  $\beta_{lm}$  with 1, that in view of (2.32) gives error terms of the order  $O(n^{-1/2})$ ,  $n \rightarrow \infty$ . It follows from the Schwarz inequality, (2.19), and (2.30) that

$$T_{21}^{1(n)} = O(n^{-1/2}), \quad n \rightarrow \infty,$$

and from (2.19), (2.32) that

$$\begin{aligned} T_{21}^{2(n)} &\leq n^{-3/2} \|U\| \cdot \|(U_{11}, \dots, U_{nn})^T\| \cdot \|((U A^{(n)} U)_{11}, \dots, (U A^{(n)} U)_{nn})^T\| \\ &= O(n^{-1/2}), \quad n \rightarrow \infty. \end{aligned}$$

This and (4.5) yield

$$|\mathbf{E}\{(T_{21}^{1(n)} + T_{21}^{2(n)})\xi_n^{A^\circ}\}| \leq cn^{-1/2} \mathbf{Var}\{\xi_n^A\}^{1/2} = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (4.38)$$

We also have

$$T_{21}^{3(n)} = O(1), \quad n \rightarrow \infty. \quad (4.39)$$

Let us show that

$$\mathbf{E}\{T_{21}^{3(n)}\xi_n^{A^\circ}\} = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (4.40)$$

For this purpose consider

$$R_n = n^{-3/2} \sum_{l,m=1}^n \mathbf{E}\{U_{ll}(t_1)U_{mm}(t_2)(U A^{(n)} U)_{lm}\xi_n^{A^\circ}\}.$$

Putting here  $U_{jj} = \mathbf{E}\{U_{jj}\} + U_{jj}^\circ$ ,  $j = l, m$ , and using (6.7) we get

$$\begin{aligned} R_n &= v(t_1)v(t_2)n^{-3/2} \sum_{l,m=1}^n \mathbf{E}\{(U A^{(n)} U)_{lm}\xi_n^{A^\circ}\} \\ &\quad + v(t_1)n^{-3/2} \sum_{l,m=1}^n \mathbf{E}\{U_{mm}^\circ(t_2)(U A^{(n)} U)_{lm}\xi_n^{A^\circ}\} \\ &\quad + n^{-3/2} \sum_{l,m=1}^n \mathbf{E}\{U_{ll}^\circ(t_1)U_{mm}(t_2)(U A^{(n)} U)_{lm}\xi_n^{A^\circ}\} + o(1), \quad n \rightarrow \infty. \end{aligned} \quad (4.41)$$

It follows from the Schwarz inequality, (6.7) and (6.8) that the first term in the r.h.s. of (4.41) is of the order  $O(n^{-1/2})$ ,  $n \rightarrow \infty$ . We also have in view of (1.2) and (2.19)

$$\begin{aligned} n^{-3/2} \left| \sum_{l,m=1}^n U_{mm}^\circ(t_2) (U A^{(n)} U)_{lm} \right| &\leq n^{-1} \|U A U\| \cdot \|(U_{11}^\circ, \dots, U_{nn}^\circ)^T\| \\ &\leq n^{-1/2} \left( \sum_{m=1}^n |U_{mm}^\circ(t_2)|^2 \right)^{1/2}. \end{aligned}$$

Hence, by the Schwarz inequality and (6.7)

$$\begin{aligned} \left| n^{-3/2} \sum_{l,m=1}^n \mathbf{E} \{ U_{mm}^\circ(t_2) (U A^{(n)} U)_{lm} \xi_n^{A^\circ} \} \right| \\ \leq n^{-1/2} \left( \sum_{m=1}^n \mathbf{Var} \{ U_{mm}(t_2) \} \right)^{1/2} \mathbf{Var} \{ \xi_n^A \}^{1/2} = O(n^{-1/2}), \quad n \rightarrow \infty. \end{aligned}$$

Thus, the second and third terms in the r.h.s. of (4.41) are of the order  $O(n^{-1/2})$ ,  $n \rightarrow \infty$ , and we get (4.40). Now (4.38) – (4.40) yield for  $T_{21}^{(n)}$  of (4.34):

$$T_{21}^{(n)} = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (4.42)$$

Applying (2.24) – (2.27) to calculate  $T_{22}^{(n)}$  and  $T_{23}^{(n)}$  of (4.34) we get terms of the form

$$\begin{aligned} n^{-3/2} \sum_{l,m=1}^n U_{lm} (U A^{(n)} U)_{lm} (U A^{(n)} U)_{lm}, \\ n^{-3/2} \sum_{l,m=1}^n U_{ll} (U A^{(n)} U)_{mm} (U A^{(n)} U)_{lm}, \end{aligned} \quad (4.43)$$

where as it follows from the Schwarz inequality and (2.31) – (2.32) the first term is of the order  $O(n^{-1/2})$ , and the second is of the order  $O(1)$ ,  $n \rightarrow \infty$ . Hence, we are left with

$$\begin{aligned} T_{22}^{(n)} + T_{23}^{(n)} &= -\frac{1}{n^{3/2}} \sum_{l,m=1}^n \mathbf{E} \{ 2(U_{ll} * (U * A^{(n)} U))_{mm}(t_1) (U * C^{(n)} U)_{lm}(t_2) \\ &\quad + ((U * C^{(n)} U))_{lm}(t_1) (U_{mm} * (U * C^{(n)} U)_{ll})(t_2) / 2 \} \\ &\quad + O(n^{-1/2}), \quad n \rightarrow \infty. \end{aligned} \quad (4.44)$$

Now it follows from (4.34), (4.42), (4.44), and (6.13) that

$$\lim_{n \rightarrow \infty} T_2^{(n)}(t_1, t_2) = \kappa_3 \lim_{n \rightarrow \infty} (T_{22}^{(n)} + T_{23}^{(n)})(t_1, t_2) = -\kappa_3 [2T_2(t_1, t_2) + T_2(t_2, t_1)], \quad (4.45)$$

where

$$T_2(t_1, t_2) = [(K_A^{(1)} - K_A^{(2)})(v * v * v)(t_1) + K_A^{(2)}(v * tv)(t_1)] \cdot (v * v)(t_2) \quad (4.46)$$

with  $K_A^{(1)}$ ,  $K_A^{(2)}$  of (4.7) – (4.8) and  $v$  of (2.42). We also have

$$(v * v)(t) = -iw^{-2} \int_{-2w}^{2w} e^{i\mu t} \mu \rho_{sc}(\mu) d\mu, \quad (4.47)$$

$$(v * tv)(t) = w^{-2} \int_{-2w}^{2w} e^{i\mu t} \left[ 1 - \frac{2w^2}{4w^2 - \mu^2} \right] \rho_{sc}(\mu) d\mu, \quad (4.48)$$

$$(v * v * v)(t) = w^{-4} \int_{-2w}^{2w} e^{it\mu} (w^2 - \mu^2) \rho_{sc}(\mu) d\mu. \quad (4.49)$$

Putting (4.47) – (4.49) in (4.46), then plugging the result in the l.h.s. of (4.31) we get after some calculations (4.31).

Consider now  $T_3^{(n)}$  of (4.29):

$$T_3^{(n)} = \frac{1}{6n^2} \sum_{l,m=1}^n \kappa_{4,lm} \mathbf{E} \{ D_{lm}^3 ((U * A^{(n)} U)_{ml}(t_1, s) \xi_n^{A_0}(t_2, s)) \},$$

where in view of (2.7) and (4.13)

$$\kappa_{4,lm} = \kappa_4 - 9\delta_{lm} w^4.$$

It follows from (2.32) and (4.5) that in (4.50) we can replace  $\kappa_{4,lm}$  with  $\kappa_4$ , which gives error terms of the order  $O(n^{-1/2})$ ,  $n \rightarrow \infty$ , and write

$$\begin{aligned} T_3^{(n)} &= \frac{\kappa_4}{6n^2} \sum_{l,m=1}^n \mathbf{E} \{ \xi_n^{A_0} \cdot D_{lm}^3 (U * A^{(n)} U)_{ml} + 3D_{lm} \xi_n^A \cdot D_{lm}^2 (U * A^{(n)} U)_{ml} \\ &\quad + 3D_{lm}^2 \xi_n^A \cdot D_{lm} (U * A^{(n)} U)_{ml} + (U * A^{(n)} U)_{ml} \cdot D_{lm}^3 \xi_n^A \}, \\ &=: \kappa_4 [T_{31}^{(n)} + T_{32}^{(n)} + T_{33}^{(n)} + T_{34}^{(n)}] + O(n^{-1/2}), \quad n \rightarrow \infty. \end{aligned} \quad (4.50)$$

Treating  $T_{31}^{(n)}$  similar to  $T_{21}^{(n)}$  of (4.34) (see (4.40) – (4.42)) one can get

$$T_{31}^{(n)} = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (4.51)$$

Besides, it can be shown with the help of (2.30) – (2.32) and (4.5) that all terms containing off-diagonal entries  $U_{lm}$  or  $(U A^{(n)} U)_{lm}$  vanish in the limit  $n \rightarrow \infty$ , hence,

$$T_{32}^{(n)} + T_{34}^{(n)} = O(n^{-1/2}),$$

and we are left with

$$\begin{aligned} T_3^{(n)} &= -\frac{i\kappa_4}{n^2} \sum_{l,m=1}^n \mathbf{E} \{ (U_{ll} * (U * A^{(n)} U)_{mm})(t_1) \\ &\quad \times (U_{ll} * (U * C^{(n)} U)_{mm} + U_{mm} * (U * C^{(n)} U)_{ll})(t_2) \} + O(n^{-1/2}), \end{aligned}$$

as  $n \rightarrow \infty$ . Now it follows from (6.11) that

$$\lim_{n \rightarrow \infty} T_3^{(n)} = -2i\kappa_4 [K_A^{(3)}(v * v * v)(t_1)(v * v * v)(t_2) + 2T_A^2(v * tv)(t_1)(v * tv)(t_2)].$$

This and (4.48) – (4.49) yield after some calculations (4.32).

It remains to show (4.33). It is much simpler because in this case we have additional factors  $n^{-1/2}$  (see (4.29)), so that treating  $T_j$ ,  $j = 4, 5, 6$  similar to  $T_j$ ,  $j = 2, 3$  one can easily get (4.33). This completes the proof of the lemma. ■



## 5 Limiting probability law for $\xi_n^A[\varphi]$

**Theorem 5.1** Consider the real symmetric Wigner random matrix of the form

$$M^{(n)} = n^{-1/2}W^{(n)}, \quad W^{(n)} = \{W_{jk} \in \mathbb{R}, W_{jk} = W_{kj} = (1 + \delta_{jk})^{1/2}V_{jk}\}_{j,k=1}^n, \quad (5.1)$$

where  $\{V_{jk}\}_{1 \leq j \leq k < \infty}$  are i.i.d. random variables such that

$$\mathbf{E}\{V_{11}\} = 0, \quad \mathbf{E}\{V_{11}^2\} = w^2,$$

and functions  $\ln \mathbf{E}\{e^{itV_{11}}\}$  and  $\mathbf{E}\{e^{it|V_{11}|}\}$  are entire.

Let  $\{A^{(n)}\}_{n=1}^\infty$  satisfies (1.2) – (1.3),  $C^{(n)} = A^{(n)} + A^{(n)T}$ , and there exist

$$A_p = \lim_{n \rightarrow \infty} n^{-p/2} \left( \sum_{l,m=1}^n (C_{lm}^{(n)})^p + (2^{(2-p)/2} - 1) \sum_{m=1}^n (C_{mm}^{(n)})^p \right) / 2, \quad p \geq 3. \quad (5.2)$$

Then for any  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , whose Fourier transform (3.1) satisfies (4.3), the random variable  $\xi_n^{A^\circ}[\varphi]$  converges in distribution as  $n \rightarrow \infty$  to the random variable  $\xi^A[\varphi]$  such that

$$\ln \mathbf{E}\{e^{ix\xi^A[\varphi]}\} = -x^2 V_W[\varphi] / 2 + \sum_{p=3}^{\infty} \frac{\kappa_p A_p}{p!} (ix^*)^p, \quad (5.3)$$

where

$$x^* = \frac{x}{w^2} \int_{-2w}^{2w} \varphi(\mu) \mu \rho_{sc}(\mu) d\mu, \quad (5.4)$$

$\rho_{sc}$  is the density of the semicircle law (2.43), and  $V_W[\varphi]$  is given by (4.14).

**Remark 5.2** It can be shown that in the case of matrix  $\widetilde{M}^{(n)} = n^{-1/2}V^{(n)}$ , the Theorem 5.1 holds true with

$$\ln \mathbf{E}\{e^{ix\xi^A[\varphi]}\} = -V_{\widetilde{W}}^1[\varphi] x^2 / 2 + \sum_{p=3}^{\infty} \frac{\kappa_p \widetilde{A}_p}{p!} (ix^*)^p,$$

where  $V_{\widetilde{W}}^1[\varphi]$  is given by (4.19) with  $w_2 = 1$ , and

$$\widetilde{A}_p = \lim_{n \rightarrow \infty} n^{-p/2} \left( \sum_{l,m=1}^n (C_{lm}^{(n)})^p + (2^{(-2p+1)/2} - 1) \sum_{m=1}^n (C_{mm}^{(n)})^p \right) / 2.$$

**Remark 5.3** In the case of matrix elements (see (1.5))  $A_p = 2^{p/2}$ , and we obtain the result of [26] (see Theorem 3.4).

In the case of bilinear forms (see (1.6) – (1.7)) we have for  $A_p$  of (5.2):

$$A_p = \lim_{n \rightarrow \infty} \left( \left( \sum_{l=1}^n (\eta_l^{(n)})^p \right)^2 + (2^{(2-p)/2} - 1) \sum_{l=1}^n (\eta_l^{(n)})^{2p} \right), \quad p \geq 3.$$

In particular, if  $\eta_m^{(n)} = O(n^{-1/2})$ ,  $n \rightarrow \infty$  for all  $m = 1, \dots, n$ , then  $A_p = 0$ ,  $p \geq 3$ , and the random variable  $(\varphi(M^{(n)})^\circ \eta^{(n)}, \eta^{(n)})$  converges in distribution to the Gaussian random variable with zero mean and the variance  $V_{GOE}^{(M\eta, \eta)}[\varphi]$  of (3.20).

**Remark 5.4** It follows from Theorem 5.1 that if  $\varphi$  is even, then the random variable  $\xi_n^{A^\circ}[\varphi]$  converges in distribution to the Gaussian random variable with zero mean and the variance  $V_{GOE}[\varphi] + C_{\kappa_4}[\varphi, \varphi]$  (see (4.11) – (4.14)).

**Proof.** Note that in view of (2.21) and (5.1) we can write

$$W_{lm} = \beta_{lm}^{-1/2} V_{lm}. \quad (5.5)$$

Besides, since  $\ln \mathbf{E}\{e^{itV_{11}}\}$  is entire then we have

$$\sum_{p=1}^{\infty} \frac{x^p |\kappa_{p+1}|}{p!} < \infty, \quad \forall x > 0, \quad (5.6)$$

where  $\kappa_p$  is the  $p$ th cumulant of  $V_{11}$ .

Consider the characteristic functions

$$Z_n^A(x) = \mathbf{E} \left\{ e^{ix(\xi_n^A[\varphi])^\circ} \right\} \quad (5.7)$$

and

$$\widehat{Z}_n^A(x) = \mathbf{E} \left\{ e^{ix(\widehat{\xi}_n^A[\varphi])^\circ} \right\}, \quad (5.8)$$

where  $\widehat{\xi}_n^A[\varphi]$  corresponds to the GOE matrix  $\widehat{M}^{(n)} = n^{-1/2} \widehat{W}^{(n)}$  (2.3). In view of Theorem 3.3, (4.14), and (5.3) it suffices to prove that for any  $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \ln Z_n^A(x) / \widehat{Z}_n^A(x) = -(C_{\kappa_3}[\varphi, \varphi] + C_{\kappa_4}[\varphi, \varphi])x^2/2 + \sum_{p=3}^{\infty} \frac{\kappa_p A_p}{p!} (ix^*)^p. \quad (5.9)$$

Following the idea of the proof of Theorem 3.2 we introduce the "interpolating" random matrix  $M^{(n)}(s)$  (see (4.22)), put

$$Z_n^A(x, s) = \mathbf{E} \{ e_n(x, s) \}, \quad e_n(x, s) = e^{ix(\xi_n^{A,s}[\varphi])^\circ}, \quad (5.10)$$

$$\xi_n^{A,s}[\varphi] = \text{Tr } \varphi(M^{(n)}(s))A^{(n)},$$

$$\xi_n^{A,s}(t) = \text{Tr } U(t, s)A^{(n)}, \quad U(t, s) = e^{itM(s)}, \quad (5.11)$$

and write

$$\begin{aligned} \ln Z_n^A(x) / \widehat{Z}_n^A(x) &= \int_0^1 \frac{\partial}{\partial s} \ln Z_n^A(x, s) ds \\ &= -\frac{x}{2} \int_0^1 \frac{ds}{Z_n^A(x, s)} \int \left( \frac{1}{\sqrt{ns}} \sum_{l,m=1}^n \mathbf{E} \{ W_{lm}^{(n)} \Psi_{lm} \} \right. \\ &\quad \left. - \frac{1}{\sqrt{n(1-s)}} \sum_{l,m=1}^n \mathbf{E} \{ \widehat{W}_{lm} \Psi_{lm} \} \right) F[\varphi](t) dt, \end{aligned} \quad (5.12)$$

where

$$\Psi_{lm} = \Psi_{lm}(t, x, s) = (U * A^{(n)}U)_{ml}(t, s) e_n^\circ(x, s). \quad (5.13)$$

(cf (4.24) – (4.26)). Let us note that unlike functions  $\Phi_{lm}$  of (4.26), having all derivatives  $D_{lm}^p \Phi_{lm}$  of the order  $O(n^{3/2})$  (see (4.27)), here we have  $D_{lm}^p \Psi_{lm} = O(n^{(p+1)/2})$ , and there is no such finite  $p \in \mathbb{N}$  that  $\varepsilon_p$  of (2.8) vanishes as  $n \rightarrow \infty$ . Hence, instead of (2.8), used while treating (4.24), here for every term of the first sum of the r.h.s. of (5.12) we apply infinite version of (2.8) given by (2.10) (see also (2.16)). To do this we check first that  $\Psi_{lm}(x, t)$  satisfies condition (2.11). Assume that the Fourier transform (3.1) of  $\varphi$  satisfies

$$\int |F[\varphi](t)| |t|^l dt < C_\varphi l! \quad \forall l \in \mathbb{N}, \quad (5.14)$$

where  $C_\varphi$  is an absolute constant. Using the Leibnitz rule we obtain

$$D_{lm}^p \Psi_{lm}(x, t, s) = \sum_{q=0}^p \binom{p}{q} D_{lm}^{p-q} (U * A^{(n)} U)_{ml}(t, s) D_{lm}^q e_n^\circ(x, s), \quad (5.15)$$

where

$$D_{lm}^q e_n(x, s) = ix D_{lm}^{q-1} (e_n(x, s) D_{lm} \xi_n^{A,s}[\varphi]), \quad (5.16)$$

(see (5.10)), so that

$$D_{lm}^q e_n(x, s) = e_n(x, s) \sum_{r=1}^q (ix)^r \sum_{\substack{\bar{q} = (q_1, \dots, q_r) : \\ q_1 + \dots + q_r = q}} C_{\bar{q}, r} \prod_{t=1}^r D_{lm}^{q_t} \xi_n^{A,s}[\varphi],$$

and

$$\sum_{\bar{q}, r} C_{\bar{q}, r} \leq 2^q.$$

Hence,

$$|D_{lm}^q e_n(x, s)| \leq (2(1 + |x|))^q \max_{1 \leq r \leq q, \sum_{t=1}^r q_t = q} \prod_{t=1}^r |D_{lm}^{q_t} \xi_n^{A,s}[\varphi]|,$$

where

$$D_{lm}^q \xi_n^{A,s}[\varphi] = \int F[\varphi](\theta) D_{lm}^q \xi_n^{A,s}(\theta) d\theta \quad (5.17)$$

with  $\xi_n^{A,s}$  of (5.11), and in view of (2.24), (2.34) and (5.14)

$$|D_{lm}^q \xi_n^{A,s}[\varphi]| \leq \int |F[\varphi](\theta)| |D_{lm}^q \xi_n^{A,s}(\theta)| d\theta \leq C_A C_\varphi 2^{q+1}, \quad (5.18)$$

so that

$$|D_{lm}^q e_n(x, s)| \leq (c\sqrt{n}(1 + |x|))^q. \quad (5.19)$$

Here and in what follows  $c$  depends only on  $A$  and  $\varphi$ . This, (2.30), and (5.15) yield

$$|D_{lm}^p \Psi_{lm}(x, t, s)| \leq (c\sqrt{n}(1 + |x| + t))^{p+1}, \quad x \in \mathbb{R}, \quad t > 0. \quad (5.20)$$

Thus,  $\Psi_{lm}$  for every  $x \in \mathbb{R}$ ,  $t > 0$  satisfies (2.11). Besides, for every  $x \in \mathbb{R}$ ,  $t > 0$  (2.12) follows from (5.6). Now applying differentiation formula (2.10) with  $\zeta = W_{lm}^{(n)}$  and  $\Phi = \Psi_{lm}$

to every term of the first sum and differentiation formula (2.4) to every term of the second sum in the r.h.s. of (5.12) and taking in account (5.5), we get (see also (2.10)):

$$\ln Z_n^A(x)/\widehat{Z}_n^A(x) = -\frac{x}{2} \int_0^1 \frac{ds}{Z_n^A(x,s)} \int \sum_{p=2}^{\infty} s^{(p-1)/2} \frac{\kappa_{p+1}}{p!} S_p^{(n)}(x,t,s) F[\varphi](t) dt, \quad (5.21)$$

where

$$S_p^{(n)}(x,t,s) = \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p+1)/2} \mathbf{E}\{D_{lm}^p \Psi_{lm}(x,t,s)\}. \quad (5.22)$$

It was shown in [26] that in the case of matrix elements (1.5) the series in (5.21) converges uniformly in  $n \in \mathbb{N}$ ,  $(t,x) \in K$  for any compact set  $K \subset \{(x,t) \in \mathbb{R}^2 : t > 0\}$ . In general case the proof is almost the same with the obvious modifications. It is based on (5.6), the estimate

$$A_p \leq 2^{p/2}, \quad \forall p \in \mathbb{N}, \quad (5.23)$$

following from (5.2) and (1.2), and on uniform bound

$$|S_p^{(n)}(x,t,s)| \leq (C_K)^l, \quad \forall (t,x) \in K, \quad n \in \mathbb{N}, \quad s \in [0,1], \quad (5.24)$$

which can be obtained with the help of (2.23) – (2.34). Here  $C_K$  is an absolute constant depending only on  $K$ . In view of the uniform convergence of the series, to make the limiting transition as  $n \rightarrow \infty$  in (5.21) it suffices to find the limits

$$S_p = \lim_{n \rightarrow \infty} S_p^{(n)}$$

for every fixed  $p \in \mathbb{N}$ . We have

$$\begin{aligned} S_p^{(n)} &= \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p+1)/2} \mathbf{E}\left\{ (U * AU)_{lm} D_{lm}^p e_n^\circ + p D_{lm} (U * AU)_{lm} D_{lm}^{p-1} e_n^\circ \right. \\ &\quad + \frac{p(p-1)}{2} D_{lm}^2 (U * AU)_{lm} D_{lm}^{p-2} e_n^\circ + (1 - \delta_{p2}) \frac{p(p-1)(p-2)}{6} D_{lm}^3 (U * AU)_{lm} D_{lm}^{p-3} e_n^\circ \\ &\quad \left. + (1 - \delta_{p2})(1 - \delta_{p3}) \sum_{q=0}^{p-4} \binom{p}{q} D_{lm}^{p-q} (U * A^{(n)} U)_{ml} D_{lm}^q e_n^\circ \right\} \\ &= S_{p1}^{(n)} + S_{p2}^{(n)} + \frac{p(p-1)}{2} S_{p3}^{(n)} + (1 - \delta_{p2}) \frac{p(p-1)(p-2)}{6} S_{p4}^{(n)} + (1 - \delta_{p2})(1 - \delta_{p3}) S_{p5}^{(n)}. \end{aligned} \quad (5.25)$$

It follows from (2.30) – (2.32) and (5.19) that

$$S_{p5}^{(n)} = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (5.26)$$

Since

$$\begin{aligned} D_{lm}^q e_n(x,s) &= e_n(x,s) (ix D_{lm} \xi_n^{A,s}[\varphi])^q + O(n^{(q-1)/2}) \\ &= e_n(x,s) \left( -x \beta_{lm} \int \widehat{\varphi}(\theta) (U * C^{(n)} U)_{lm}(\theta) d\theta \right)^q + O(n^{(q-1)/2}), \quad n \rightarrow \infty, \end{aligned} \quad (5.27)$$

then

$$S_{p4}^{(n)} = \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p+1)/2} \mathbf{E}\{D_{lm}^3(U * A^{(n)}U)_{lm} \times (-x\beta_{lm} \int \widehat{\varphi}(\theta)(U * C^{(n)}U)_{lm}(\theta)d\theta)^{p-3} e_n\} + O(n^{-1/2}), \quad p > 3,$$

and by (2.30) – (2.31)  $S_{p4}^{(n)} = O(n^{-1/2})$ ,  $p > 3$ . If  $p = 3$ , then

$$S_{34}^{(n)} = \frac{1}{n^2} \sum_{l,m=1}^n \beta_{lm}^{-2} \mathbf{E}\{D_{lm}^3(U * A^{(n)}U)_{lm} e_n^\circ(x, s)\},$$

(compare with  $T_{31}^n$  of (4.50)), and in addition to (2.30) – (2.31) we use (6.10) to show that  $S_{34}^{(n)} = O(n^{-1/2})$ . Thus,

$$S_{p4}^{(n)} = O(n^{-1/2}), \quad n \rightarrow \infty, \quad p \geq 3. \quad (5.28)$$

Consider now  $S_{p3}^{(n)}$  of (5.25):

$$\begin{aligned} S_{p3}^{(n)} &= \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p+1)/2} \mathbf{E}\{D_{lm}^2(U * A^{(n)}U)_{lm} D_{lm}^{p-2} e_n^\circ\} \\ &= \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{(p-5)/2} \mathbf{E}\{D_{lm}^2(U * A^{(n)}U)_{lm} (-x \int \widehat{\varphi}(\theta)(U * C^{(n)}U)_{lm}(\theta)d\theta)^{p-2} e_n\} \\ &\quad + O(n^{-1/2}), \quad n \rightarrow \infty, \quad p > 2, \end{aligned}$$

where we used (5.27). There arise sums of three types

$$\begin{aligned} S_{p3}^{1(n)} &= \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{(p-1)/2} U_{ll} U_{mm} (U A^{(n)} U)_{lm} (U C^{(n)} U)_{lm}^{p-2}, \\ S_{p3}^{2(n)} &= \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{(p-1)/2} U_{lm} U_{lm} (U A^{(n)} U)_{lm} (U C^{(n)} U)_{lm}^{p-2}, \\ S_{p3}^{3(n)} &= \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{(p-1)/2} U_{ll} U_{lm} (U A^{(n)} U)_{mm} (U C^{(n)} U)_{lm}^{p-2}, \end{aligned}$$

where we omit arguments of  $U$  and put  $(U C^{(n)} U)_{lm}^q = \prod_{j=1}^q (U(t_{j1}) C^{(n)} U(t_{j2}))_{lm}$ . If  $p = 2$ , then treating  $S_{23}^{(n)}$  similar to  $T_{21}^{(n)}$  of (4.34) (see (4.34) – (4.42)) we get  $S_{23}^{(n)} = O(n^{-1/2})$ ,  $n \rightarrow \infty$ . In case  $p > 2$  we use following from (2.29) – (2.32) asymptotic relations

$$\sum_{l,m=1}^n |U_{lm}| |(U A^{(n)} U)_{lm}| = O(n), \quad (5.29)$$

$$\sum_{l,m=1}^n |(U A^{(n)} U)_{lm}| |(U A^{(n)} U)_{lm}| = O(n), \quad (5.30)$$

$$\sum_{l,m=1}^n |(U A^{(n)} U)_{mm}| |(U A^{(n)} U)_{lm}| = O(n\sqrt{n}), \quad (5.31)$$

$$\sum_{m=1}^n |(U A^{(n)} U)_{mm}| |(U A^{(n)} U)_{mm}| = O(n), \quad (5.32)$$

as  $n \rightarrow \infty$ . They together with (2.30) allows to show that  $S_{p3}^{j(n)}$ ,  $j = 1, 2, 3$  are of the order  $O(n^{-1/2})$ ,  $n \rightarrow \infty$ , so that

$$S_{p3}^{(n)} = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (5.33)$$

Consider  $S_{p2}^{(n)}$  of (5.25):

$$\begin{aligned} S_{p2}^{(n)} &= \frac{p}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p+1)/2} \mathbf{E}\{D_{lm}(U * A^{(n)}U)_{lm}(t, s) D_{lm}^{p-1} e_n^\circ(x, s)\} \\ &= -\frac{2px}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p-1)/2} \mathbf{E}\{(U_{ll} * (U * A^{(n)}U)_{mm} + U_{lm} * (U * A^{(n)}U)_{lm})(t, s) \\ &\quad \times D_{lm}^{p-2}(e_n(x, s) D_{lm} \xi_n^{A,s}[\varphi])\} \\ &= S_{p2}^{1(n)} + S_{p2}^{2(n)}, \end{aligned} \quad (5.34)$$

where we used (2.27) and (5.16). Since

$$\begin{aligned} D_{lm}^q(e_n(x, s) D_{lm} \xi_n^{A,s}[\varphi]) &= D_{lm}^q e_n(x, s) \cdot D_{lm} \xi_n^{A,s}[\varphi] \\ &\quad + q D_{lm}^{q-1} e_n(x, s) \cdot D_{lm}^2 \xi_n^{A,s}[\varphi] + O(n^{(q-1)/2}), \quad n \rightarrow \infty, \end{aligned} \quad (5.35)$$

where

$$\begin{aligned} D_{lm} \xi_n^{A,s}[\varphi] &= i \beta_{lm} \int (U *^{(n)} C U)_{lm}(\theta, s) F[\varphi](\theta) d\theta, \\ D_{lm}^2 \xi_n^{A,s}[\varphi] &= -\beta_{lm}^2 \int (U_{ll} * (U * C^{(n)}U)_{mm} + U_{mm} * (U * C^{(n)}U)_{ll} \\ &\quad + 2U_{lm} * (U * C^{(n)}U)_{lm})(\theta, s) F[\varphi](\theta) d\theta, \end{aligned} \quad (5.36)$$

then putting (5.35) with  $q = p - 2$  in  $S_{p2}^{2(n)}$  of (5.34) and applying (5.19), (5.30), and (5.32) we get

$$S_{p2}^{2(n)} = O(n^{-1/2}), \quad n \rightarrow \infty, \quad (5.37)$$

and

$$\begin{aligned} S_{p2}^{1(n)} &= -2px \int F[\varphi](\theta) d\theta \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p+1)/2} \mathbf{E}\{(U_{ll} * (U * A^{(n)}U)_{mm})(t, s) \\ &\quad \times [D_{lm}^{p-2} e_n(x, s) \cdot i(U * C^{(n)}U)_{lm}(\theta, s) \\ &\quad - (p-2) D_{lm}^{p-3} e_n(x, s) \cdot (U_{ll} * (U * C^{(n)}U)_{mm} + U_{mm} * (U * C^{(n)}U)_{ll})(\theta, s)]\} \\ &\quad + O(n^{-1/2}), \quad n \rightarrow \infty. \end{aligned}$$

It follows from (5.27) and (5.29) – (5.32) that  $S_{p2}^{1(n)}$  does not vanish only if  $p = 2$  or  $p = 3$ , so that putting  $e_n(x, s) = Z_n^A(x, s) + e_n^\circ(x, s)$  and using (6.10) and (6.12), we get

$$\begin{aligned} S_{p2}^{1(n)} &= x Z_n^A(x, s) \int \left[ -\frac{4i\delta_{p2}}{n^{3/2}} \sum_{l,m=1}^n \mathbf{E}\{(U_{ll} * (U * A^{(n)}U)_{mm})(t, s) (U * C^{(n)}U)_{lm}(\theta, s)\} \right. \\ &\quad + \frac{6\delta_{p3}}{n^2} \sum_{l,m=1}^n \mathbf{E}\{(U_{ll} * (U * A^{(n)}U)_{mm})(t, s) (U_{ll} * (U * C^{(n)}U)_{mm} \\ &\quad \left. + U_{mm} * (U * C^{(n)}U)_{ll})(\theta, s)\} \right] F[\varphi](\theta) d\theta + O(n^{-1/2}), \end{aligned} \quad (5.38)$$

as  $n \rightarrow \infty$ . Such expressions were considered while proving Theorem 4.2 (see Lemma 4.6). Treating  $S_{p2}^{1(n)}$ ,  $p = 2, 3$  in the same way and using (6.11), (6.13) and (4.47) – (4.49), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} -\frac{x}{2} \int_0^1 \frac{ds}{Z_n^A(x, s)} \int \left[ \frac{\kappa_3 \sqrt{s}}{2} S_{22}^{1(n)}(x, t, s) + \frac{\kappa_4 s}{6} S_{32}^{1(n)}(x, t, s) \right] F[\varphi](t) dt \\ = -\left( \frac{2}{3} C_{\kappa_3}[\varphi, \varphi] + C_{\kappa_4}[\varphi, \varphi] \right) x^2/2 \end{aligned} \quad (5.39)$$

with  $C_{\kappa_3}[\varphi, \varphi]$ ,  $C_{\kappa_4}[\varphi, \varphi]$  of (4.11) – (4.12) (see also (5.9), (5.21)).

At last consider  $S_{p1}^{(n)}$  of (5.25):

$$\begin{aligned} S_{p1}^{(n)} &= \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p+1)/2} \mathbf{E}\{(U * A^{(n)}U)_{lm}(t, s) D_{lm}^p e_n^\circ(x, s)\} \\ &= \frac{ix}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p+1)/2} \mathbf{E}\{(U * A^{(n)}U)_{lm}(t, s) D_{lm}^{p-1}(e_n(x, s) D_{lm} \xi_n^{A,s}[\varphi])\} \\ &= \frac{ix}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{-(p+1)/2} \mathbf{E}\{(U * A^{(n)}U)_{lm}(t, s) [D_{lm}^{p-1} e_n(x, s) \cdot D_{lm} \xi_n^{A,s}[\varphi] \\ &\quad + (p-1) D_{lm}^{p-2} e_n(x, s) \cdot D_{lm}^2 \xi_n^{A,s}[\varphi]]\} + O(n^{-1/2}) \\ &= S_{p1}^{1(n)} + S_{p1}^{2(n)} + O(n^{-1/2}), \quad n \rightarrow \infty, \end{aligned} \quad (5.40)$$

where we used (5.16), (5.35), and then (2.34), (5.19), and (5.29) – (5.32) to estimate the vanishing term. It follows from (5.27) and (5.30) – (5.32) that if  $p > 2$ , then

$$S_{p1}^{2(n)} = O(n^{-1/2}), \quad n \rightarrow \infty, \quad p > 2. \quad (5.41)$$

If  $p = 2$ , then

$$\begin{aligned} S_{21}^{2(n)} &= -ix \int \frac{1}{n^{3/2}} \sum_{l,m=1}^n \mathbf{E}\{e_n(x, s) (U * A^{(n)}U)_{lm}(t, s) (U_{ll} * (U * C^{(n)}U)_{mm} \\ &\quad + U_{mm} * (U * C^{(n)}U)_{ll})(\theta, s)\} F[\varphi](\theta) d\theta + O(n^{-1/2}), \end{aligned}$$

and similar to (5.39)

$$\lim_{n \rightarrow \infty} -\frac{x}{2} \int_0^1 \frac{ds}{Z_n^A(x, s)} \int \frac{\kappa_3 \sqrt{s}}{2} S_{21}^{2(n)}(x, t, s) F[\varphi](t) dt = -\left( \frac{1}{3} C_{\kappa_3}[\varphi, \varphi] \right) x^2/2. \quad (5.42)$$

Using (5.27) with  $q = p - 1$  and (5.36) we get for  $S_{p1}^{1(n)}$  of (5.40):

$$\begin{aligned} S_{p1}^{1(n)} &= \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \beta_{lm}^{(p-1)/2} \mathbf{E}\{(U * A^{(n)}U)_{lm}(t, s) e_n(x, s) \\ &\quad \times (-x \int \widehat{\varphi}(\theta) (U * C^{(n)}U)_{mm}(\theta) d\theta)^p\} + O(n^{-1/2}), \quad n \rightarrow \infty, \end{aligned} \quad (5.43)$$

where we estimate the vanishing term with the help of (5.30) and (5.32). Putting here

$$\beta_{lm}^{(p-1)/2} = 1 + \delta_{lm}(2^{(1-p)/2} - 1)$$

and  $e_n(x, s) = Z_n^A(x, s) + e_n^\circ(x, s)$ , and then applying first parts of (6.14) – (6.15), we get

$$\begin{aligned} S_{p1}^{1(n)} &= \frac{Z_n^A(x, s)}{n^{(p+1)/2}} \sum_{l,m=1}^n \mathbf{E}\{(U * A^{(n)}U)_{lm}(t, s) \left(-x \int \widehat{\varphi}(\theta)(U * C^{(n)}U)_{lm}(\theta)d\theta\right)^p\} \\ &\quad + 2(2^{(1-p)/2} - 1) \frac{Z_n^A(x, s)}{n^{(p+1)/2}} \sum_{m=1}^n \mathbf{E}\{(U * A^{(n)}U)_{mm}(t, s) \\ &\quad \times \left(-x \int \widehat{\varphi}(\theta)(U * C^{(n)}U)_{mm}(\theta)d\theta\right)^p\} + O(n^{-1/2}), \quad n \rightarrow \infty. \end{aligned}$$

This and second parts of (6.14) – (6.15) yield for  $p \geq 2$

$$\lim_{n \rightarrow \infty} -\frac{x}{2} \frac{\kappa_{p+1}}{p!} \int_0^1 \frac{ds}{Z_n^A(x, s)} \int s^{(p-1)/2} S_{p1}^{1(n)}(x, t, s) F[\varphi](t) dt = \frac{\kappa_{p+1} A_{p+1}}{(p+1)!} (ix^*)^{p+1} \quad (5.44)$$

with  $A_p$  and  $x^*$  defined in (5.2) and (5.4). Now putting (5.39), (5.42), and (5.44) in (5.12) we get (5.44) and finish the proof of the theorem under condition (5.44).

The case of  $\varphi \in E = \{\psi : \int (1 + |t|)^4 |\widehat{\psi}(t)| dt < \infty\}$  can be obtained via a standard approximation procedure. Indeed, since the set  $D = \{\varphi : \int |\widehat{\varphi}(t)| |t|^l dt < C_\varphi l!, \forall l \in \mathbb{N}\}$  is big enough (in particular, it contains functions  $e^{-x^2} P_m(x)$ , where  $P_m(x)$  is a polynomial), then for any  $\varphi \in E$  there exists a sequence  $\{\varphi_k\} \subset D$ , such that

$$\lim_{k \rightarrow \infty} \int_{-2w}^{2w} |\varphi(\lambda) - \varphi_k(\lambda)| d\lambda = 0. \quad (5.45)$$

Denote for the moment the characteristic functions of  $\xi_n^A[\varphi]$  and  $\xi^A[\varphi]$  as  $Z_n[\varphi]$  and  $Z[\varphi]$ , to make explicit their dependence on  $\varphi$ . We have then for any  $\varphi \in E$

$$\begin{aligned} |Z_n[\varphi] - Z[\varphi]| &\leq |Z_n[\varphi] - Z_n[\varphi_k]| + |Z_n[\varphi_k] - Z[\varphi_k]| + |Z[\varphi_k] - Z[\varphi]| \\ &:= T_{nk}^{(1)} + T_{nk}^{(2)} + T_{nk}^{(3)}. \end{aligned} \quad (5.46)$$

The second term of the r.h.s. vanishes after the limit  $n \rightarrow \infty$  in view of the above proof, since  $\varphi_k \in D$ . For the first term we have from (5.8) and the Schwarz inequality that

$$|T_{nk}^{(1)}| = |\mathbf{E}\{e^{ix\xi_n^{A^\circ}[\varphi]} - e^{ix\xi_n^{A^\circ}[\varphi_k]}\}| \leq |x| (n \mathbf{Var}\{\xi_n^A[\varphi_k]\})^{1/2}, \quad \psi_k = \varphi - \varphi_k,$$

and then Theorem 4.2 implies that

$$\limsup_{n \rightarrow \infty} |T_{nk}^{(1)}| \leq |x| (V_W[\psi_k])^{1/2}.$$

Since  $V_W$  of (4.14) is continuous with respect to the  $L^1$  convergence, then in view of (5.45)  $T_{nk}^{(1)}$  vanishes after the subsequent limits  $n \rightarrow \infty$ ,  $k \rightarrow \infty$ .

At last, we have by (5.6), (5.23), and the continuity of the r.h.s. of (5.3) with respect to the  $L^1$  convergence, that the third term of (5.46) vanishes after the limit  $k \rightarrow \infty$ . Thus, we proved Theorem under condition (4.3). ■



## 6 Auxiliary results

**Lemma 6.1** Consider matrix  $A^{(n)}$ , satisfying (1.2) – (1.3),  $C^{(n)} = A^{(n)} + A^{(n)T}$ , and a unitary matrix

$$U(t) = U^{(n)}(t) = e^{itM^{(n)}},$$

corresponding to the Wigner matrix  $M^{(n)}$  of (2.1) – (2.2). Denote

$$\begin{aligned} U^j &= U(t_j), \\ \overline{t^{(p)}} &= (t_1, \dots, t_p), \end{aligned}$$

and define

$$\begin{aligned} \xi_n^A(t) &= \text{Tr } A^{(n)} U(t), \\ \eta_n^A(t_1, t_2) &= n^{-3/2} \sum_{l,m=1}^n (U^1 A^{(n)} U^2)_{lm}, \end{aligned} \tag{6.1}$$

$$v_n^I(t_1, t_2, t_3) = n^{-1} \sum_{m=1}^n U_{mm}^1 (U^2 A^{(n)} U^3)_{mm}, \tag{6.2}$$

$$v_n^C(\overline{t^{(4)}}) = n^{-1} \sum_{m=1}^n (U^1 C^{(n)} U^2)_{mm} (U^3 A^{(n)} U^4)_{mm}, \tag{6.3}$$

$$\omega_n(\overline{t^{(5)}}) = n^{-3/2} \sum_{l,m=1}^n U_{ll}^1 (U^2 A^{(n)} U^3)_{mm} (U^4 C^{(n)} U^5)_{lm}, \tag{6.4}$$

$$\gamma_n^{(1)}(\overline{t^{(2p+2)}}) = n^{-(p+1)/2} \sum_{l,m=1}^n (U^1 A^{(n)} U^2)_{lm} \prod_{j=2}^{p+1} (U^{2j-1} C^{(n)} U^{2j})_{lm}, \quad p \geq 2, \tag{6.5}$$

$$\gamma_n^{(2)}(\overline{t^{(2p+2)}}) = n^{-(p+1)/2} \sum_{m=1}^n (U^1 A^{(n)} U^2)_{mm} \prod_{j=2}^{p+1} (U^{2j-1} C^{(n)} U^{2j})_{mm}, \quad p \geq 2, \tag{6.6}$$

and put

$$\overline{f} = \mathbf{E}\{f\}.$$

Then we have under conditions of Theorem 4.2:

$$(i) \mathbf{Var}\{\xi_n^A(t)\} \leq c(1 + |t|)^8, \quad \lim_{n \rightarrow \infty} \bar{\xi}_n^A(t) = T_A \cdot v(t), \quad (6.7)$$

$$(ii) \mathbf{Var}\{\eta_n^A(t_1, t_2)\} = O(n^{-1}) \quad \lim_{n \rightarrow \infty} \bar{\eta}_n^A(t_1, t_2) = K_A'^{(2)} \cdot v(t_1)v(t_2), \quad (6.8)$$

$$(iii) \mathbf{Var}\{v_n^I(t_1, t_2, t_3)\} = O(n^{-1}), \quad \lim_{n \rightarrow \infty} \bar{v}_n^I(t_1, t_2, t_3) = T_A \cdot v(t_1)v(t_2 + t_3), \quad (6.9)$$

$$(iv) \mathbf{Var}\{v_n^C(\overline{t^{(4)}})\} = O(n^{-1}), \quad (6.10)$$

$$\lim_{n \rightarrow \infty} \bar{v}_n^C(\overline{t^{(4)}}) = 2K_A^{(3)} \prod_{j=1}^4 v(t_j) + 2T_A^2 \cdot v(t_1 + t_2)v(t_3 + t_4), \quad (6.11)$$

$$(v) \mathbf{Var}\{\omega_n(\overline{t^{(5)}})\} = O(n^{-1/2}), \quad (6.12)$$

$$\lim_{n \rightarrow \infty} \bar{\omega}_n(\overline{t^{(5)}}) = (K_A^{(1)} - K_A^{(2)}) \prod_{j=1}^5 v(t_j) + K_A^{(2)} v(t_1)v(t_4)v(t_5)v(t_2 + t_3), \quad (6.13)$$

$$(vi) \mathbf{Var}\{\gamma_n^{(1)}(\overline{t^{(2p+2)}})\} = O(n^{-1/2}), \quad \lim_{n \rightarrow \infty} \bar{\gamma}_n^{(1)}(\overline{t^{(2p+2)}}) = K_A^{(4)} \prod_{j=1}^{p+1} v(t_j), \quad (6.14)$$

$$(vii) \mathbf{Var}\{\gamma_n^{(2)}(\overline{t^{(2p+2)}})\} = O(n^{-1/2}), \quad \lim_{n \rightarrow \infty} \bar{\gamma}_n^{(2)}(\overline{t^{(2p+2)}}) = K_A^{(5)} \prod_{j=1}^{p+1} v(t_j), \quad (6.15)$$

where  $O(n^\alpha)$ ,  $n \rightarrow \infty$ , can depend on  $\overline{t^{(p)}}$ ,  $v$  is defined in (2.42),  $K_A^{(j)}$ ,  $j = 1, 2, 3$  are defined in (4.7) – (4.9), and

$$K_A'^{(2)} = \lim_{n \rightarrow \infty} n^{-3/2} \sum_{l,m=1}^n A_{lm}^{(n)}, \quad (6.16)$$

$$K_A^{(4)} = \lim_{n \rightarrow \infty} n^{-(p+1)/2} \sum_{l,m=1}^n A_{lm}^{(n)} (C_{lm}^{(n)})^p, \quad (6.17)$$

$$K_A^{(5)} = \lim_{n \rightarrow \infty} n^{-(p+1)/2} \sum_{m=1}^n A_{mm}^{(n)} (C_{mm}^{(n)})^p. \quad (6.18)$$

**Remark 6.2** All statements of the lemma remain valid under conditions of Theorem 5.1.

**Proof. GOE case.** Firstly we prove the lemma supposing that matrix  $M^{(n)}$  belongs to the GOE. Statement (i) in GOE case was proved in Lemma 2.3.

(ii) We have by Poincaré inequality (2.5)

$$\mathbf{Var}\{\eta_n^A(t_1, t_2)\} \leq \frac{w^2}{n^4} \sum_{1 \leq j \leq k \leq n} \beta_{jk}^{-1} \mathbf{E}\left\{ \left| D_{jk} \sum_{l,m=1}^n (U^1 A^{(n)} U^2)_{lm} \right|^2 \right\}.$$

This and (2.26) show that it suffices to estimate

$$T_n = \frac{1}{n^4} \sum_{j,k=1}^n \left| \sum_{l,m=1}^n U_{lj}^1 (U^2 A^{(n)} U^3)_{km} \right|^2.$$

We have

$$\begin{aligned} T_n &= \frac{1}{n^4} \sum_{j,k=1}^n \sum_{l,l',m,m'=1}^n U_{lj}^1 \bar{U}_{j'l'}^1 (\bar{U}^3 A^{(n)T} \bar{U}^2)_{mk} (U^2 A^{(n)} U^3)_{km} \\ &= \frac{1}{n^3} \sum_{m,m'=1}^n (\bar{U}^3 A^{(n)T} A^{(n)} U^3)_{mm'} = \frac{1}{n^3} \left| \sum_{m,p=1}^n (A^{(n)} U^3)_{pm} \right|^2 \leq \frac{1}{n^2} \text{Tr } A A^{(n)T} = O(n^{-1}), \end{aligned}$$

hence,

$$\mathbf{Var}\{\eta_n^A(t_1, t_2)\} = O(n^{-1}), \quad n \rightarrow \infty.$$

Now applying Duhamel formula (2.17) and differentiation formulas (2.4), (2.27), and then estimating the error terms with the help of (3.12), one can get

$$\begin{aligned} \bar{\eta}_n^A(t_1, t_2) &= n^{-3/2} \sum_{l,m=1}^n \mathbf{E}\{(A^{(n)} U(t_2))_{lm}\} \\ &\quad - w^2 \int_0^{t_1} dt_3 \int_0^{t_3} \bar{v}_n(t_3 - t_4) \bar{\eta}_n^A(t_3, t_2) dt_4 + o(1), \quad n \rightarrow \infty, \end{aligned} \quad (6.19)$$

where by (3.7) and (2.42)

$$\lim_{n \rightarrow \infty} n^{-3/2} \sum_{l,m=1}^n \mathbf{E}\{(A^{(n)} U(t_2))_{lm}\} = \lim_{n \rightarrow \infty} \bar{v}_n(t_2) n^{-3/2} \sum_{l,m=1}^n A_{lm}^{(n)} = K_A'^{(2)} v(t_2)$$

with  $K_A'^{(2)}$  of (6.16). Thus, we have for  $\eta^A = \lim_{n \rightarrow \infty} \bar{\eta}_n^A$ :

$$\eta^A(t_1, t_2) + w^2 \int_0^{t_1} dt_3 \int_0^{t_3} v(t_3 - t_4) \eta^A(t_4, t_2) dt_4 = K_A'^{(2)} v(t_1),$$

and by (2.47)

$$\eta^A(t_1, t_2) = K_A'^{(2)} v(t_1) v(t_2), \quad (6.20)$$

so (6.8) is proved.

(iii) Putting  $U_{mm} = U_{mm}^\circ + \bar{U}_{mm}$  and using (3.7) we get

$$\bar{v}_n^I(t_1, t_2, t_3) = \bar{v}_n(t_1) \bar{\xi}_n^A(t_2 + t_3) + \bar{r}_n, \quad r_n = n^{-1} \sum_{m=1}^n (U_{mm}^1)^\circ (U^2 A^{(n)} U^3)_{mm}. \quad (6.21)$$

By the Schwarz inequality and (2.32)

$$\begin{aligned} |r_n| &\leq \left( \sum_{m=1}^n |(U^2 A^{(n)} U^3)_{mm}|^2 \right)^{1/2} \left( \sum_{m=1}^n |(U_{mm}^1)^\circ|^2 \right)^{1/2} \\ &= O(n^{-1/2}) \left( \sum_{m=1}^n |(U_{mm}^1)^\circ|^2 \right)^{1/2}. \end{aligned} \quad (6.22)$$

It follows from (3.12) that

$$\mathbf{E}\left\{\sum_{m=1}^n |(U_{mm}^1)^\circ|^2\right\} = O(1), \quad n \rightarrow \infty. \quad (6.23)$$

This, (6.22), and the Schwarz inequality for expectations yield

$$|\bar{r}_n| = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (6.24)$$

Now (6.7), (6.21), and (6.24) give

$$\lim_{n \rightarrow \infty} \bar{v}_n^I(t_1, t_2, t_3) = T_A \cdot v(t_1)v(t_2 + t_3).$$

Besides, we have for  $V_n = \mathbf{Var}\{v_n^I\}$ :

$$V_n = \mathbf{E}\left\{n^{-1} \sum_{m=1}^n U_{mm}^1 (U^2 A^{(n)} U^3)_{mm} \cdot v_n^{I^\circ}\right\} = \bar{v}_n(t_1) \mathbf{E}\{n^{-1} \xi_n^A(t_2 + t_3) v_n^{I^\circ}\} + \mathbf{E}\{r_n v_n^{I^\circ}\}$$

with  $r_n$  of (6.21). It follows from the Schwarz inequality, (3.12) and (6.22) – (6.24), that

$$V_n \leq O(n^{-1/2}) V_n^{1/2}, \quad n \rightarrow \infty.$$

This proves (6.9).

(iv) The proof of (6.10) repeats with the obvious modifications that one of (6.8). Let us prove (6.11). Applying Duhamel formula (2.17), differentiation formulas (2.4), (2.23) – (2.27), and then estimating the error terms with the help of (3.12), one can get (cf (3.36) and (6.19))

$$\begin{aligned} n^{-1} \sum_{m=1}^n \mathbf{E}\{(U^1 A^{(n)} U^2)_{mm} (U^3 C^{(n)} U^4)_{mm}\} &= n^{-1} \sum_{m=1}^n \mathbf{E}\{(A^{(n)} U^2)_{mm} (U^3 C^{(n)} U^4)_{mm}\} \quad (6.25) \\ &\quad - w^2 \int_0^{t_1} dt_5 \int_0^{t_5} \mathbf{E}\left\{v_n(t_5 - t_6) n^{-1} \sum_{m=1}^n (U^6 A^{(n)} U^2)_{mm} (U^3 C^{(n)} U^4)_{mm}\right\} dt_6 \\ &\quad - w^2 \int_0^{t_1} dt_5 \int_0^{t_5} \mathbf{E}\left\{n^{-1} \xi_n^A(t_5 + t_6) n^{-1} \sum_{m=1}^n U_{mm}(t_2 - t_6) (U^3 C^{(n)} U^4)_{mm}\right\} dt_6 \\ &\quad + o(1), \quad n \rightarrow \infty, \end{aligned}$$

and

$$\begin{aligned} n^{-1} \sum_{m=1}^n \mathbf{E}\{(A^{(n)} U^2)_{mm} (U^3 C^{(n)} U^4)_{mm}\} &= n^{-1} \sum_{m=1}^n A_{mm}^{(n)} \mathbf{E}\{(U^3 C^{(n)} U^4)_{mm}\} \quad (6.26) \\ &\quad - w^2 \int_0^{t_2} dt_5 \int_0^{t_5} \mathbf{E}\left\{v_n(t_5 - t_6) n^{-1} \sum_{m=1}^n (A^{(n)} U^6)_{mm} (U^3 C^{(n)} U^4)_{mm}\right\} dt_6 \\ &\quad + o(1), \quad n \rightarrow \infty, \end{aligned}$$

and also

$$\begin{aligned}
n^{-1} \sum_{m=1}^n A_{mm}^{(n)} \mathbf{E}\{(U^3 C^{(n)} U^4)_{mm}\} &= n^{-1} \sum_{m=1}^n A_{mm}^{(n)} \mathbf{E}\{(C^{(n)} U^4)_{mm}\} \\
&- w^2 \int_0^{t_3} dt_5 \int_0^{t_5} \mathbf{E}\left\{v_n(t_5 - t_6) n^{-1} \sum_{m=1}^n A_{mm}^{(n)} (U^6 C^{(n)} U^4)_{mm}\right\} dt_6 \\
&- w^2 \int_0^{t_3} dt_5 \int_0^{t_4} \mathbf{E}\left\{n^{-1} \xi_n^C(t_5 + t_6) n^{-1} \sum_{m=1}^n A_{mm}^{(n)} U_{mm}(t_4 - t_6)\right\} dt_6 \\
&+ o(1), \quad n \rightarrow \infty,
\end{aligned} \tag{6.27}$$

where by (3.7)

$$n^{-1} \sum_{m=1}^n A_{mm}^{(n)} \mathbf{E}\{(C^{(n)} U^4)_{mm}\} = \bar{v}_n(t_4) n^{-1} \sum_{m=1}^n A_{mm}^{(n)} C_{mm}^{(n)},$$

so that

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{m=1}^n A_{mm}^{(n)} \mathbf{E}\{(C^{(n)} U^4)_{mm}\} = 2D_A \cdot v(t_4), \quad D_A = \lim_{n \rightarrow \infty} n^{-1} \sum_{m=1}^n (A_{mm}^{(n)})^2. \tag{6.28}$$

Denote

$$\begin{aligned}
v^C(\overline{t^{(4)}}) &= \lim_{n \rightarrow \infty} \bar{v}_n^C(\overline{t^{(4)}}), \\
G(t_2, t_3, t_4) &= \lim_{n \rightarrow \infty} n^{-1} \sum_{m=1}^n \mathbf{E}\{(A^{(n)} U^2)_{mm} (U^3 C^{(n)} U^4)_{mm}\}, \\
H(t_3, t_4) &= \lim_{n \rightarrow \infty} n^{-1} \sum_{m=1}^n A_{mm}^{(n)} \mathbf{E}\{(U^3 C^{(n)} U^4)_{mm}\}.
\end{aligned}$$

(More accurately, it can be shown that there exist corresponding convergent subsequences. But all these subsequences have the same limits, which are unique solutions of the system of integral equations below. Hence, we can write limits of whole sequences.) It follows from (6.25) – (6.28), (6.7) and (6.9) that  $v^C$ ,  $G$ , and  $H$  satisfy the system of integral equations:

$$\begin{aligned}
v^C(\overline{t^{(4)}}) &+ w^2 \int_0^{t_1} dt_5 \int_0^{t_5} v(t_5 - t_6) v^C(t_6, t_2, t_3, t_4) dt_6 \\
&= G(t_2, t_3, t_4) - 2w^2 T_A^2 v(t_3 + t_4) \int_0^{t_1} dt_5 \int_0^{t_5} v(t_5 + t_6) v(t_2 - t_6) dt_6, \\
G(t_2, t_3, t_4) &+ w^2 \int_0^{t_2} dt_5 \int_0^{t_5} v(t_5 - t_6) G(t_6, t_3, t_4) dt_6 = H(t_3, t_4), \\
H(t_3, t_4) &+ w^2 \int_0^{t_3} dt_5 \int_0^{t_5} v(t_5 - t_6) H(t_6, t_4) dt_6 \\
&= 2D_A \cdot v(t_4) - 2w^2 T_A^2 \int_0^{t_3} dt_5 \int_0^{t_4} v(t_5 + t_6) v(t_4 - t_6) dt_6.
\end{aligned}$$

Solving the equations with the help of (2.47) – (2.48) we get

$$\begin{aligned} H(t_3, t_4) &= 2K_A^{(3)}v(t_3)v(t_4) + 2T_A^2v(t_3 + t_4), \\ G(t_2, t_3, t_4) &= v(t_2)H(t_3, t_4), \\ v^C(\overline{t^{(4)}}) &= v(t_1)G(t_2, t_3, t_4) + 2T_A^2(v(t_1 + t_2) - v(t_1)v(t_2))v(t_3 + t_4), \end{aligned}$$

so that

$$v^C(\overline{t^{(4)}}) = 2K_A^{(3)} \prod_{j=1}^4 v(t_j) + 2T_A^2v(t_1 + t_2)v(t_3 + t_4),$$

and (6.11) is proved.

(v) Similar to (6.21) – (6.24) we have

$$\overline{\omega}_n(\overline{t^{(5)}}) = \overline{v}_n(t_1)\overline{\Gamma}_n(t_2, t_3, t_4, t_5) + \overline{r}_n, \quad (6.29)$$

where

$$\begin{aligned} \Gamma_n(t_2, t_3, t_4, t_5) &= n^{-3/2} \sum_{l,m=1}^n (U^2 A^{(n)} U^3)_{mm} (U^4 C^{(n)} U^5)_{lm}, \\ r_n(\overline{t^{(5)}}) &= n^{-3/2} \sum_{l,m=1}^n (U_{ll}^1)^\circ (U^2 A^{(n)} U^3)_{mm} (U^4 C^{(n)} U^5)_{lm}, \end{aligned} \quad (6.30)$$

and by (2.19), the Schwarz inequality, (2.32), and (6.23)

$$\begin{aligned} |\overline{r}_n| &\leq n^{-3/2} \|C^{(n)}\| \left( \mathbf{E} \left\{ \sum_{l=1}^n |(U_{ll}^1)^\circ|^2 \right\} \right)^{1/2} \left( \mathbf{E} \left\{ \sum_{m=1}^n |(U^2 A^{(n)} U^3)_{mm}|^2 \right\} \right)^{1/2} \\ &= O(n^{-1/2}), \quad n \rightarrow \infty. \end{aligned} \quad (6.31)$$

Applying Duhamel formula (2.17), differentiation formulas (2.4), (2.23) – (2.27), and then estimating the error terms with the help of (3.12), one can get for  $\Gamma_n$  (cf (6.25) – (6.27)):

$$\begin{aligned} \overline{\Gamma}_n(t_2, t_3, t_4, t_5) &= \overline{B}_n(t_3, t_4, t_5) \\ &\quad - w^2 \int_0^{t_2} dt_6 \int_0^{t_6} \mathbf{E} \{ v_n(t_6 - t_7) \Gamma_n(t_7, t_3, t_4, t_5) \} dt_7 \\ &\quad - w^2 \int_0^{t_2} dt_6 \int_0^{t_3} \mathbf{E} \{ n^{-1} \xi_n^A(t_6 + t_7) D_n(t_3 - t_7, t_4, t_5) \} dt_7 \\ &\quad + o(1), \quad n \rightarrow \infty, \end{aligned}$$

where

$$\begin{aligned} B_n(t_3, t_4, t_5) &= n^{-3/2} \sum_{l,m=1}^n (A^{(n)} U^3)_{mm} (U^2 C^{(n)} U^3)_{lm}, \\ D_n(\tau, t_4, t_5) &= n^{-3/2} \sum_{l,m=1}^n U_{mm}(\tau) (U^2 C^{(n)} U^3)_{lm}. \end{aligned}$$

Similar to (6.29) – (6.31) it can be shown that

$$\overline{D}_n(\tau, t_4, t_5) = \overline{v}_n(\tau) \overline{\eta}_n^C(t_4, t_5) + O(n^{-1/2}), \quad n \rightarrow \infty,$$

where  $\eta_n^C$  is defined in (6.1), so that by (6.8)

$$\lim_{n \rightarrow \infty} \overline{D}_n(\tau, t_4, t_5) = 2K_A'^{(2)} v(\tau) v(t_4) v(t_5). \quad (6.32)$$

We also have for  $B_n$ :

$$\overline{B}_n(t_3, t_4, t_5) = \overline{v}_n(t_3) n^{-3/2} \sum_{l,m=1}^n \mathbf{E} \left\{ A_{mm}^{(n)} (U^4 C^{(n)} U^5)_{lm} \right\} + R_n, \quad (6.33)$$

where

$$|R_n| \leq n^{-1} \|A^{(n)}\| \left( \mathbf{E} \left\{ \sum_{m=1}^n |(AU^3)_{mm}^\circ|^2 \right\} \right)^{1/2}.$$

By the standard argument based on Poincaré inequality (2.5) one can easily get

$$\mathbf{Var}\{(AU)_{mm}(t)\} \leq C|t|^2 n^{-1} (AA^{(n)T})_{mm},$$

hence,

$$R_n = O(n^{-1/2}), \quad n \rightarrow \infty. \quad (6.34)$$

Besides, repeating with obvious modifications steps leading from (6.19) to (6.20), we get

$$\lim_{n \rightarrow \infty} n^{-3/2} \sum_{l,m=1}^n \mathbf{E} \left\{ A_{mm}^{(n)} (U^4 C^{(n)} U^5)_{lm} \right\} = K_A^{(1)} v(t_4) v(t_5).$$

This, (6.33), and (6.34) yield

$$B(t_3, t_4, t_5) := \lim_{n \rightarrow \infty} \overline{B}_n(t_3, t_4, t_5) = K_A^{(1)} v(t_3) v(t_4) v(t_5). \quad (6.35)$$

Plugging (6.32), (6.35) in (6.30) we get equation with respect to  $\Gamma = \lim_{n \rightarrow \infty} \overline{\Gamma}_n$ :

$$\begin{aligned} \Gamma(t_2, t_3, t_4, t_5) + w^2 \int_0^{t_2} dt_6 \int_0^{t_6} v(t_6 - t_7) \Gamma(t_7, t_3, t_4, t_5) dt_7 \\ = B(t_3, t_4, t_5) - w^2 K_A^{(2)} v(t_4) v(t_5) \int_0^{t_2} dt_6 \int_0^{t_3} v(t_6 + t_7) v(t_3 - t_7) dt_7, \end{aligned}$$

where we put  $K_A^{(2)} = 2T_A K_A'^{(2)}$  (see (4.8), (6.16)). Solving the equation with the help of (2.47) – (2.48), we obtain

$$\begin{aligned} \Gamma(t_2, t_3, t_4, t_5) &= v(t_2) B(t_3, t_4, t_5) + K_A^{(2)} (v(t_2 + t_3) - v(t_2) v(t_3)) v(t_4) v(t_5) \\ &= (K_A^{(1)} - K_A^{(2)}) \prod_{j=1}^5 v(t_j) + K_A^{(2)} v(t_2 + t_3) v(t_4) v(t_5). \end{aligned}$$

This and (6.29) – (6.31) finally yield (6.13).

(vi) It follows from Poincaré inequality (2.5) that

$$\mathbf{Var}\{\gamma_n^{(1)}(\overline{t^{(2p+2)}})\} \leq \frac{w^2}{n^{(p+2)}} \sum_{1 \leq j \leq k \leq n} \beta_{jk}^{-1} \mathbf{E}\left\{\left|D_{jk} \sum_{l,m=1}^n (U^1 A^{(n)} U^2)_{lm} \prod_{j=2}^{p+1} (U^{2j-1} C^{(n)} U^{2j})_{lm}\right|^2\right\}.$$

Taking into account (2.26) we see that to get the first part of (6.14) it suffices to show that

$$R_n := \frac{1}{n^{(p+2)}} \sum_{j,k=1}^n \left| \sum_{l,m=1}^n U_{jl}^0 (U^1 A^{(n)} U^2)_{km} \prod_{j=2}^{p+1} (U^{2j-1} C^{(n)} U^{2j})_{lm} \right|^2 = O(n^{-1}),$$

as  $n \rightarrow \infty$ , (here  $U^0 = U(t_0)$ ). Since by (2.19)

$$\sum_{j=1}^n U_{jl} \overline{U_{j' l'}} = \delta_{ll'}, \quad \sum_{k=1}^n (U^2 A^{(n)T} U^1)_{mk} \overline{(U^1 A^{(n)} U^2)_{km'}} = \sum_{k=1}^n (U^2 A^{(n)T})_{mk} \overline{(A^{(n)} U^2)_{km'}},$$

then

$$\begin{aligned} R_n &= \frac{1}{n^{(p+2)}} \sum_{k,l=1}^n \left| \sum_{m=1}^n (A^{(n)} U^2)_{km} \prod_{j=2}^{p+1} (U^{2j-1} C^{(n)} U^{2j})_{lm} \right|^2 \\ &\leq \frac{1}{n^{(p+2)}} \sum_{k,l=1}^n \sum_{m=1}^n |(A^{(n)} U^2)_{km}|^2 \sum_{m'=1}^n \left| \prod_{j=2}^{p+1} (U^{2j-1} C^{(n)} U^{2j})_{lm'} \right|^2. \end{aligned}$$

We have by (1.2) and (2.19)

$$\sum_{k,m=1}^n |(A^{(n)} U^2)_{km}|^2 = \text{Tr } A^{(n)} A^{(n)T} = O(n), \quad n \rightarrow \infty,$$

and by (2.30) – (2.31)

$$\sum_{l,m'=1}^n \left| \prod_{j=2}^{p+1} (U^{2j-1} C^{(n)} U^{2j})_{lm'} \right|^2 = O(n^{p-1}) \sum_{l,m'=1}^n \left| (U^3 C^{(n)} U^4)_{lm'} \right|^2 = O(n^p), \quad n \rightarrow \infty.$$

Hence,  $R_n = O(n^{-1})$ ,  $n \rightarrow \infty$ , and

$$\mathbf{Var}\{\gamma_n^{(1)}(\overline{t^{(2p+2)}})\} = O(n^{-1}), \quad n \rightarrow \infty. \quad (6.36)$$

To prove (6.14) we show that every  $U(t)$  in

$$\bar{\gamma}_n^{(1)}(\overline{t^{(2p+2)}}) = n^{-(p+1)/2} \sum_{l,m=1}^n \mathbf{E}\left\{(U(t_1) A^{(n)} U(t_2))_{lm} \prod_{j=2}^{p+1} (U(t_{2j-1}) C^{(n)} U(t_{2j}))_{lm}\right\} \quad (6.37)$$

can be replaced with  $\bar{v}_n$  with the error term that vanishes as  $n \rightarrow \infty$ . For this purpose it suffices to show that

$$\bar{\gamma}_n^{(1)}(\overline{t^{(p)}}) = \bar{v}_n(t_1) \bar{\delta}_n(t_2, \dots, t_{2p+2}) + o(1), \quad n \rightarrow \infty, \quad (6.38)$$



where

$$\delta_n(t_2, \dots, t_{2p+2}) = n^{-(p+1)/2} \sum_{l,m=1}^n (A^{(n)}U(t_2))_{lm} \prod_{j=2}^{p+1} (U(t_{2j-1})C^{(n)}U(t_{2j}))_{lm}. \quad (6.39)$$

Applying Duhamel formula (2.17) and then differentiation formulas (2.4), (2.23) – (2.27), we get:

$$\begin{aligned} \bar{\gamma}_n^{(1)}(\overline{t^{(p)}}) &= \bar{\delta}_n(t_2, \dots, t_{2p+2}) \\ &\quad - w^2 \int_0^{t_1} d\tau_1 \int_0^{\tau_1} v(\tau_1 - \tau_2) \bar{\gamma}_n^{(1)}(\tau_2, t_2, \dots, t_{2p+2}) d\tau_2 \\ &\quad - w^2 \int_0^{t_1} R_n(\tau_1, t_2, \dots, t_{2p+2}) d\tau_1, \end{aligned}$$

where

$$\begin{aligned} R_n(\tau_1, t_2, \dots, t_{2p+2}) &= \int_0^{\tau_1} \mathbf{E} \left\{ v_n^\circ(\tau_2) \gamma_n^{(1)}(\tau_1 - \tau_2, t_2, \dots, t_{2p+2}) \right\} d\tau_2 \\ &\quad + \int_0^{\tau_1} (\bar{v}_n(\tau_2) - v(\tau_2)) \bar{\gamma}_n^{(1)}(\tau_1 - \tau_2, t_2, \dots, t_{2p+2}) d\tau_2 \\ &\quad + n^{-1} \tau_1 \bar{\gamma}_n^{(1)}(\tau_1, t_2, \dots, t_{2p+2}) \\ &\quad + n^{-1} \int_0^{t_2} \bar{\gamma}_n^{(1)}(\tau_1 + \tau_2, t_2 - \tau_2, \dots, t_{2p+2}) d\tau_2 \\ &\quad + \int_0^{t_2} \mathbf{E} \left\{ n^{-1} \xi_n^A(\tau_1 + \tau_2) \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n U_{lm}(t_2 - \tau_2) \prod_{j=2}^{p+1} (U(t_{2j-1})C^{(n)}U(t_{2j}))_{lm} \right\} d\tau_2 \\ &\quad + \frac{1}{n} \cdot \frac{1}{n^{(p+1)/2}} \sum_{l,m,k=1}^n \beta_{lk}^{-1} \mathbf{E} \left\{ (U(\tau_1)A^{(n)}U(t_2))_{km} D_{lk} \prod_{j=2}^{p+1} (U(t_{2j-1})C^{(n)}U(t_{2j}))_{lm} \right\}. \end{aligned} \quad (6.40)$$

It follows from (2.47) and (2.39) with  $T = -v$  and  $R'(t) = -w^2 R_n(t, t_2, \dots, t_{2p+2})$ , that

$$\bar{\gamma}_n^{(1)}(\overline{t^{(p)}}) = v(t_1) \bar{\delta}_n(t_2, \dots, t_{2p+2}) - w^2 \int_0^{t_1} v(t_1 - \tau_1) R_n(\tau_1, t_2, \dots, t_{2p+2}) d\tau_1.$$

Hence, to get (6.38) it suffices to show that

$$R_n = o(1), \quad n \rightarrow \infty. \quad (6.41)$$

Indeed, the first four terms of the r.h.s. of (6.40) vanishes because of (6.7), the fifth term is of the order  $O(n^{-1/2})$ ,  $n \rightarrow \infty$ , because of (2.30) – (2.31) and boundedness of  $n^{-1} \xi_n^A(\tau_1 + \tau_2)$ , and the last term after differentiation gives terms of the form  $n^{-1} \bar{\gamma}_n^{(1)}$  or

$$\frac{1}{n} \cdot \frac{1}{n^{(p+1)/2}} \sum_{l,m=1}^n \mathbf{E} \left\{ (UA^{(n)}UC^{(n)}U)_{lm} \prod_{j=2}^{p+1} (UC^{(n)}U)_{lm} \right\}, \quad (6.42)$$

which evidently of the order  $O(n^{-1/2})$ ,  $n \rightarrow \infty$  (see (2.30) – (2.31)). Hence, (6.41) is proved, and so does (6.38). It remains to note that (6.38) holds true for

$$\bar{\gamma}_n^{(1)}(\overline{t^{(2p+2)}}) = n^{-(p+1)/2} \sum_{l,m=1}^n \mathbf{E} \left\{ (V(t_1)A^{(n)}V(t_2))_{lm} \prod_{j=2}^{p+1} (V(t_{2j-1})C^{(n)}V(t_{2j}))_{lm} \right\}.$$

(cf (6.37)), where  $V$  is equal  $U$  or identity matrix  $I_n$ . Hence, in the limit  $n \rightarrow \infty$  we can replace all  $U$  of (6.37) with  $v$  and so get (6.14).

(vii) The proof of (vii) repeats essentially that one of (vi).

**Wigner case.** Proofs of all statements (i) – (vii) follow the same scheme based on the known facts for the GOE matrices and interpolation procedure proposed while proving Theorems 4.2 and 5.1. We demonstrate this scheme proving (i):

(i) Consider  $V_n(t) := \mathbf{Var}\{\xi_n^A(t)\}$  and note that

$$V_n(t) = \mathbf{Var}\{\hat{\xi}_n^A(t)\} + C_n^\Delta(t, -t), \quad (6.43)$$

where  $\hat{\xi}_n^A$  and  $C_n^\Delta$  are defined in (4.21) and (4.24), respectively. By (4.5) we have

$$\mathbf{Var}\{\hat{\xi}_n^A(t)\} \leq ct^2. \quad (6.44)$$

Repeating steps leading from (4.24) to (4.28) – (4.30), but using here (2.8) with  $p = 5$  instead  $p = 6$  in (4.28), we get

$$c_n^\Delta(t, -t) = \frac{i}{2} \int_0^1 \left[ \sum_{j=2}^5 s^{(j-1)/2} T_j^{(n)} + \varepsilon_5 \right] ds \quad (6.45)$$

with  $T_j^{(n)}$  of (4.29), and

$$|\varepsilon_5| \leq \frac{C_5 w_8^{7/8}}{n^{7/2}} \sum_{l,m=1}^n \sup_{M \in \mathcal{S}_n} |D_{lm}^6 \Phi_{lm}| \leq c(1 + |t|)^7. \quad (6.46)$$

Consider  $T_1^{(n)}$ . It is given by (4.34) with  $t_1 = t$ ,  $t_2 = -t$ . Since  $T_{21}^{j(n)}$ ,  $j = 1, 2, 3$  of (4) – (4.40) are bounded uniformly in  $n \in \mathbb{N}$ , and every derivative  $D_{lm}$  of  $U(t) = e^{itM^{(n)}}$  gives factor  $t$ , then

$$\left| n^{-3/2} \sum_{l,m=1}^n D_{lm}^2 (U * A^{(n)} U)_{lm} \right| \leq c(1 + |t|)^3,$$

and by the Schwarz inequality we have for  $T_{21}^{(n)}$  of (4.34):

$$|T_{21}^{(n)}| = \left| \mathbf{E} \left\{ n^{-3/2} \sum_{l,m=1}^n D_{lm}^2 (U * A^{(n)} U)_{lm} \cdot \xi_n^{A^0}(t) \right\} \right| \leq c(1 + |t|)^3 V_n^{1/2}.$$

We also have for  $T_{22}^{(n)}$  and  $T_{23}^{(n)}$  of (4.34) (see (4.44) and (2.31) – (2.32)):

$$|T_{22}^{(n)} + T_{23}^{(n)}| \leq c(1 + |t|)^3.$$

Hence,

$$|T_2^{(n)}| \leq c(1 + |t|)^3(V_n^{1/2} + 1). \quad (6.47)$$

Treating  $T_3^{(n)}$  of (4.50) and  $T_j^{(n)}$ ,  $j = 4, 5$  of (4.29) in the similar way one can get

$$|T_3^{(n)}| \leq c(1 + |t|)^4(V_n^{1/2} + 1), \quad (6.48)$$

$$|T_j^{(n)}| \leq c(1 + |t|)^{j+1}, \quad j = 4, 5. \quad (6.49)$$

Putting (6.46) – (6.49) in (6.45), and then together with (6.44) in (6.43), we get the quadratic inequality with respect to  $V_n^{1/2}$ :

$$V_n - c(1 + |t|)^4 V_n^{1/2} - c(1 + |t|)^7 \leq 0,$$

solving which we get  $V_n \leq c(1 + |t|)^4$ .

To finish the proof of (i) it remains to show that

$$\lim_{n \rightarrow \infty} n^{-1} \mathbf{E}\{\xi_n^A(t)\} = T_A v(t). \quad (6.50)$$

In the GOE case we have (see (3.8))

$$\lim_{n \rightarrow \infty} n^{-1} \mathbf{E}\{\widehat{\xi}_n^A(t)\} = T_A v(t). \quad (6.51)$$

Besides, we have

$$\begin{aligned} \xi_n^A(t) - \widehat{\xi}_n^A(t) &= \int_0^1 \frac{\partial}{\partial s} \xi_n^A(t, s) ds \\ &= \frac{i}{2} \int_0^1 \sum_{l,m=1}^n \left( \frac{1}{\sqrt{sn}} W_{lm}^{(n)} - \frac{1}{\sqrt{(1-s)n}} \widehat{W}_{lm} \right) (U * A^{(n)} U)_{lm}(t, s) ds, \end{aligned}$$

so that similar to (6.45) – (6.46)

$$n^{-1} \mathbf{E}\{\xi_n^A(t)\} - n^{-1} \mathbf{E}\{\widehat{\xi}_n^A(t)\} = \frac{i}{2} \int_0^1 \left[ s^{1/2} T_2'^{(n)} + \varepsilon_2 \right] ds,$$

where

$$T_2'^{(n)} = \frac{\kappa_3}{j! n^{5/2}} \sum_{l,m=1}^n \mathbf{E}\{D_{lm}^2(U * A^{(n)} U)_{lm}(t, s)\} = O(n^{-1}), \quad n \rightarrow \infty,$$

and

$$|\varepsilon_3| \leq \frac{C_3 \sqrt{w_8}}{n^6} \sum_{l,m=1}^n \sup_{M \in \mathcal{S}_n} |D_{lm}^4(U * A^{(n)} U)_{lm}(t, s)| = O(n^{-1/2}), \quad n \rightarrow \infty.$$

Hence,

$$n^{-1} \mathbf{E}\{\xi_n^A(t)\} - n^{-1} \mathbf{E}\{\widehat{\xi}_n^A(t)\} = O(n^{-1/2}), \quad n \rightarrow \infty.$$

This and (6.51) yield (6.50) and finish the proof of (i). ■

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